MALLA REDDY ENGINEERING COLLEGE(AUTONOMOUS)

Maisammaguda, Dhulapally, (Post & Via) Kompally, Secunderabad, Telangana-500100

Department of Electronics and Communication Engineering

B.Tech- II ECE

INSTRUCTIONAL MATIERIAL

Academic Year	: **2019-20**	**Semester**	: **II**
Course Name	: **Control Systems**		
Course Code	: **80209**	**Regulation**	: **MR-18**

Prepared by: Dr. A. Pradeep Kumar, Assistant Professor

2018-19 Onwards (MR-18)	**MALLA REDDY ENGINEERING COLLEGE** (Autonomous)	**B.Tech. IV Semester**		
Code: 80209	**CONTROL SYSTEMS** (Common to EEE and ECE)	**L**	**T**	**P**
Credits: 3		**3**	**-**	**-**

Pre-Requisites: Nil.

Course Objectives: This course introduces different ways of system representations such as Transfer function representation and state space representations and to assess the system dynamic response. It also emphasis on analysis of system performance in time and frequency domain and techniques for improving the performance.

MODULE I: Introduction   [10 Periods]

Concepts of Control Systems - Open Loop and closed loop control systems and their differences -Different examples of control systems - Classification of control systems, Feedback Characteristics, Effects of feedback, Mathematical models – Differential equations, Impulse Response and transfer functions.

Transfer Function Representation: Block diagram representation of systems considering electrical systems as examples - Block diagram algebra – Representation by Signal flow graph - Reduction using Mason's gain formula.Synchros, AC & DC servo motors and stepper motor.

MODULE II: Time Response Analysis   [9 Periods]

Standard test signals - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems - Time domain specifications – Steady state response - Steady state errors and error constants – Effects of proportional derivative, proportional integral systems.

MODULE III: Stability Analysis in S-Domain   [9 Periods]

A: The concept of stability – Routh's stability criterion – Qualitative stability and conditional stability – Limitations of Routh's stability.

B: Root Locus Technique: The root locus concept - Construction of root loci - Effects of addingpoles and zeros to G(s) H(s) on the root loci.

MODULE IV: Frequency Response Analysis   [10 Periods]

Introduction, Frequency domain specifications - Bode diagrams - Determination of frequency domain specifications and Phase margin and Gain margin - Stability analysis from Bode Plots - Polar Plots - Nyquist Plots.

Compensation Techniques: Lag, Lead and Lead -Lag Controllers design in frequencyDomain**.**

MODULE V: State Space Analysis of Continuous Systems   [10 Periods]

Concepts of state, state variables and state model, derivation of state models from block diagrams, diagonalization - Solving the Time invariant state equations - State Transition Matrix and it's Properties – Concepts of Controllability and observability.

Text Books:

1. I.J.Nagrath and M.Gopal, "Control Systems Engineering", New Age International Publishers, 5th Edition, 2007.
2. Benjamin.C. Kuo, "Automatic Control Systems", Prentice Hall of India, 7th Edition, 1995.

Reference Books:

1. A.Nagoor kani, "Control Systems", RBA Publications, 2nd Edition, 2006.
2. M.Gopal, "Control System: Principles and Design", Tata McGraw Hill, 2nd Edition, 2002.
3. Joseph J Distefano, "Schaum's Outline Series of Feedback and Control Systems", Tata McGraw Hill, 2nd Edition, 2014.
4. K. Ogata, "Modern Control Engineering", Pearson Education, New Delhi, 5th Edition, 2010.
5. M. Gopal, "Control Systems, Principles & Design", Tata McGraw Hill, 4th Edition, 2012.

E-Resources:

1. https://www.electrical4u.com/control-engineering-historical-review-and-types-of-control-engineering/
2. http://ieeecss.org/CSM/library/2011.html
3. http://nptel.ac.in/courses/108101037/

MODULE I

INTRODUCTION

Concepts of Control Systems

When a number of elements (or) components are connected in a sequence to perform a specific function, the group is thus formed is called **'System'**.

Examples − Traffic lights control system, washing machine

In a system when the output quantity is controlled by varying the input quantity is called as **'Control System'**.

Traffic lights control system is an example of control system. Here, a sequence of input signal is applied to this control system and the output is one of the three lights that will be on for some duration of time. During this time, the other two lights will be off. Based on the traffic study at a particular junction, the on and off times of the lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights control system operates on time basis.

Open Loop and closed loop control systems and their differences

Control Systems can be classified as open loop control systems and closed loop control systems based on the **feedback path**.

In **open loop control systems**, output is not fed-back to the input. So, the control action is independent of the desired output.

The following figure shows the block diagram of the open loop control system.
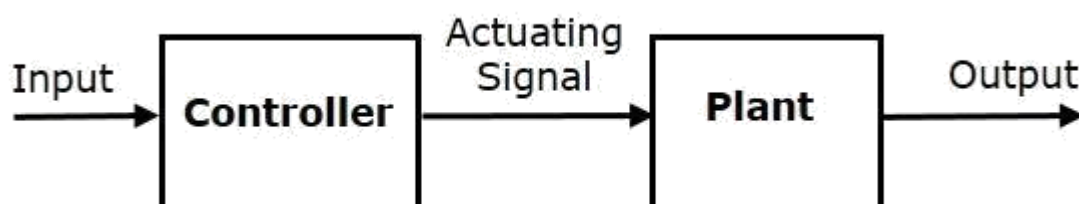
Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signal is given as an input to a plant or process which is to be controlled. So, the plant produces an output, which is controlled. The traffic lights control system which we discussed earlier is an example of an open loop control system.

In **closed loop control systems**, output is fed back to the input. So, the control action is dependent on the desired output.

The following figure shows the block diagram of negative feedback closed loop control system.
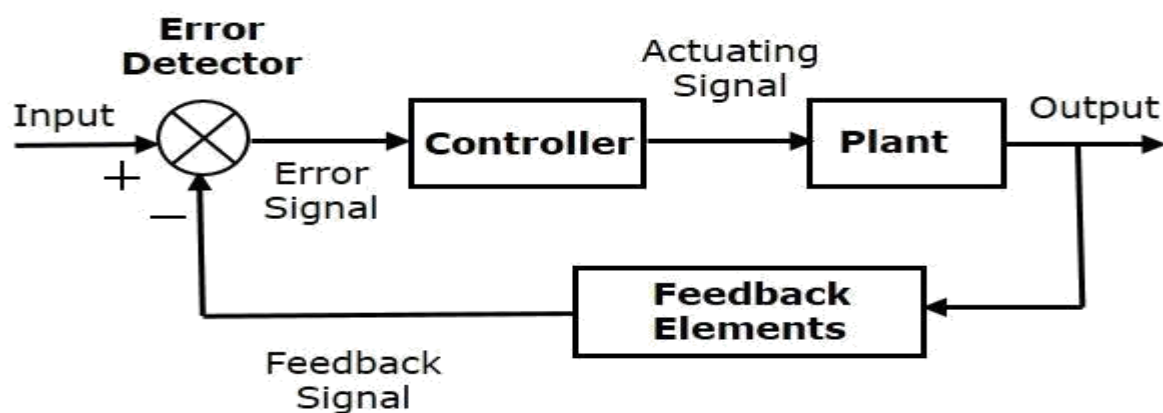
The error detector produces an error signal, which is the difference between the input and the feedback signal. This feedback signal is obtained from the block (feedback elements) by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

So, the controller produces an actuating signal which controls the plant. In this combination, the output of the control system is adjusted automatically till we get the desired response. Hence, the closed loop control systems are also called the automatic control systems. Traffic lights control system having sensor at the input is an example of a closed loop control system.

The differences between the open loop and the closed loop control systems are mentioned in the following table.

S.NO	Open loop Control system	Closed loop control system
1.	Simple and economic	Complex and costlier
2.	Consume less power	Consume more power
3.	Easy to construct with less number of components	Difficult to construct with more number of components
4.	Stability is not a major problem	Stability is a major problem

| 5.	In accurate and un reliable	Accurate and un reliable
6.	Feed back is not present	Feed back is present
7.	External disturbances, noise effects are more	External disturbances, noise effects are less
8.	Control action is independent of the desired output	Control action is dependent of the desired output

Different examples of control systems
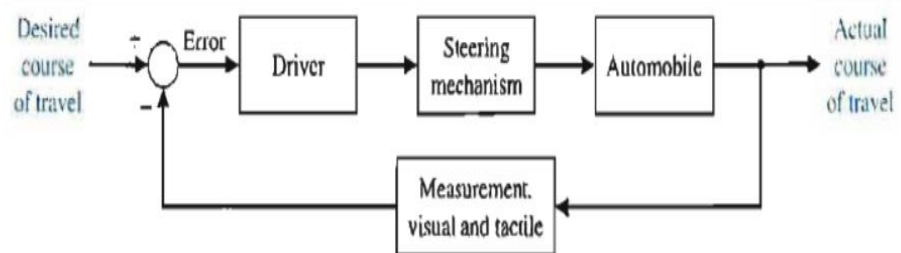

Example [1]: Automobile Steering Control System
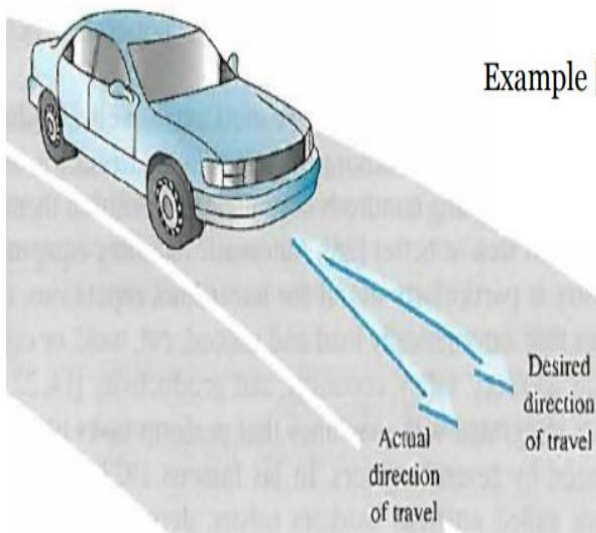
The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of steering wheel
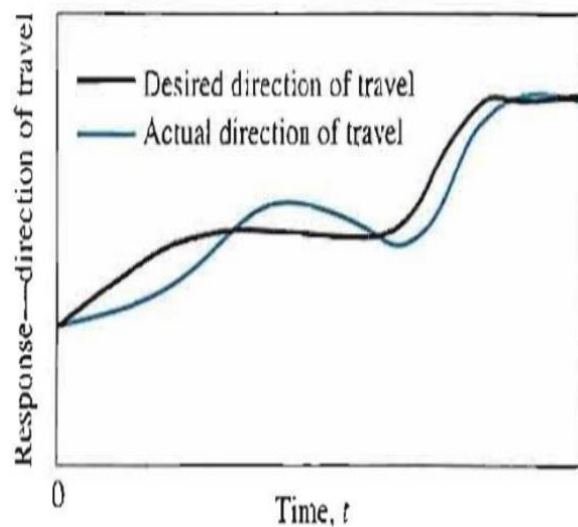
Typical direction of travel response

## Example [2]: Automobile interior cabin temperature control system

Many luxury automobiles have thermostatically controlled air-conditioning systems for the comfort of the passengers. Sketch a block diagram of an air-conditioning system where the driver sets the desired interior temperature on a dashboard panel. Identify the function of each element of the thermostatically controlled cooling system.
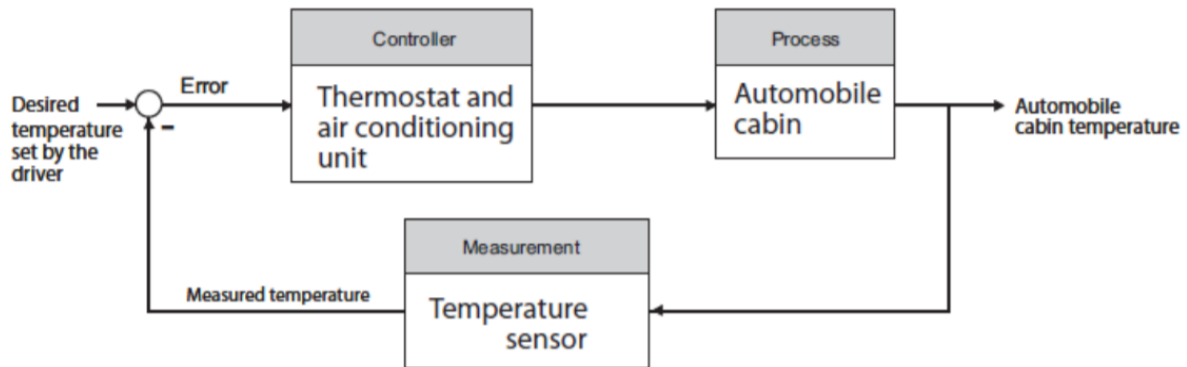
Example [3]: A car control speed system

In a car, to control speed, there is accelerator and break. Suppose break is not there, will you be able to decrease the speed when desired. Explain the need for two control signals (accelerator and break).

A control system to keep a car at a given relative position offset from a lead car:
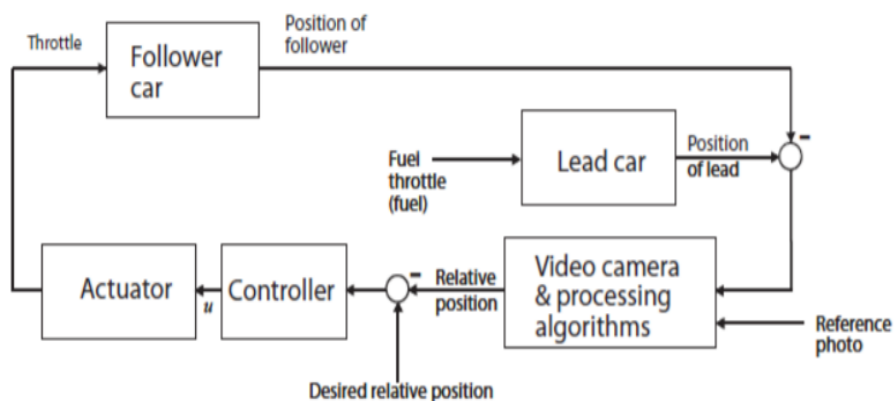
Classification of control systems

The control system may be classified in a number of ways. Some popular classifications are:

1. Depending on the hierarchy
   a. Open loop control systems
   b. Closed loop control systems

2. Depending on Presence of human being
   a. Mannual control systems
   b. Automatic control systems

3. Depending on feed back
   a. Open loop control systems
   c. Closed loop control systems

4. Depending on the main purpose of the system
   a. Temperature control systems
   b. Position control systems
   c. Velocity control systems
   d. Traffic control systems
   e. Process control systems

5. Depending on the method of analysis and design
   a. Linear control systems
   b. Non linear control systems

6. Depending on the parameter variation with time
   a. Time variant control systems
   b. Time in variant control systems

7. Depending on the types of signals used in the system
   a. Continous data control systems
   b. Discrete control systems

8. Depending on the number of inputs and outputs
   a. Single input single output systems (SISO)
   b. Multi input Multi output systems (MIMO)

9. Depending on the type of the system
   a. Type-0 system
   b. Type-1 system
   c. Type-2 system

10. Depending on the type of damping
    a. Undamped system
    b. Under damped system
    c. Critically damped system
    d. Over damped system

Feedback Characteristics

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then it is known as **feedback**. Feedback plays an important role in order to improve the performance of the control systems.

Feedback improves the system performance characteristics like stability, bandwidth, overall gain, disturbance and sensitivity. Feedback is deliberately introduced for the purpose of control the operations in systems.

There are two types of feedback −

- Positive feedback

- Negative feedback

Positive Feedback

The positive feedback adds the reference input, R(s)R(s) and feedback output. The following figure shows the block diagram of **positive feedback control system**
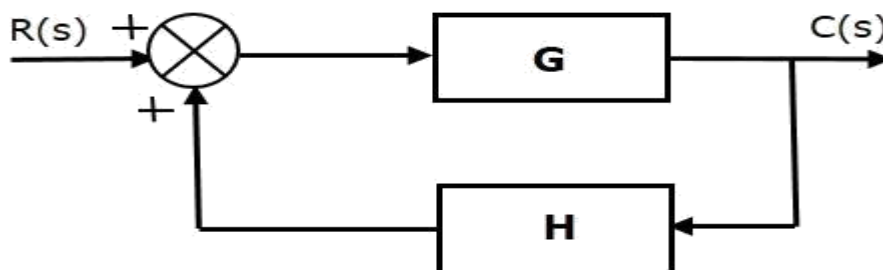
The concept of transfer function will be discussed in later chapters. For the time being, consider the transfer function of positive feedback control system is,

$$T = \frac{G}{1-GH} \qquad\qquad (\text{Equation } 1)$$
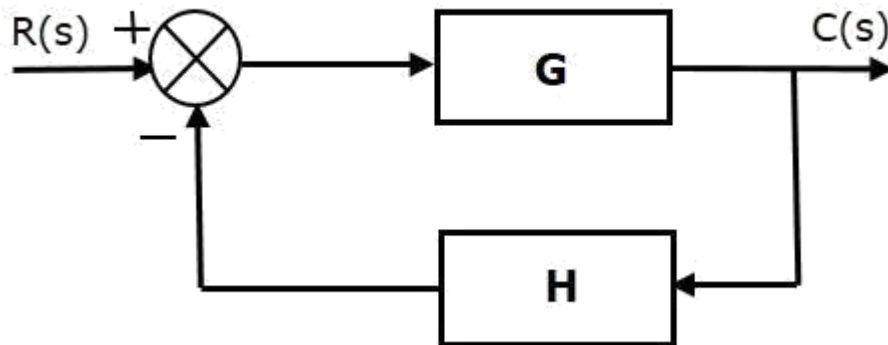
Where,

- **T** is the transfer function or overall gain of positive feedback control system.

- **G** is the open loop gain, which is function of frequency.

- **H** is the gain of feedback path, which is function of frequency.

Negative Feedback

Negative feedback reduces the error between the reference input, R(s)R(s) and system output.

The following figure shows the block diagram of the **negative feedback control system**.

Transfer function of negative feedback control system is,

$$T = \frac{G}{1+GH} \qquad\qquad \text{(Equation 2)}$$

Where,

- **T** is the transfer function or overall gain of negative feedback control system.

- **G** is the open loop gain, which is function of frequency.

- **H** is the gain of feedback path, which is function of frequency.

Effects of Feedback

Effect of Feedback on Overall Gain

- From Equation 2, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and (1+GH). So, the overall gain may increase or decrease depending on the value of (1+GH).

- If the value of (1+GH) is less than 1, then the overall gain increases. In this case, 'GH' value is negative because the gain of the feedback path is negative.

- If the value of (1+GH) is greater than 1, then the overall gain decreases. In this case, 'GH' value is positive because the gain of the feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, the feedback will increase the overall gain of the system in one frequency range and decrease in the other frequency range.

Effect of Feedback on Sensitivity

Sensitivity of the overall gain of negative feedback closed loop control system (**T**) to the variation in open loop gain (**G**) is defined as

$$S_G^T = \frac{\frac{\partial T}{T}}{\frac{\partial G}{G}} = \frac{Percentage\ change\ in\ T}{Percentage\ change\ in\ G} \qquad \text{(Equation 3)}$$

Where, ∂T is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} \qquad \text{(Equation 4)}$$

Do partial differentiation with respect to G on both sides of Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1 - G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \qquad \text{(Equation 5)}$$

From Equation 2, you will get

$$\frac{G}{T} = 1 + GH \qquad \text{(Equation 6)}$$

Substitute Equation 5 and Equation 6 in Equation 4.

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So, we got the **sensitivity** of the overall gain of closed loop control system as the reciprocal of (1+GH). So, Sensitivity may increase or decrease depending on the value of (1+GH).

- If the value of (1+GH) is less than 1, then sensitivity increases. In this case, 'GH' value is negative because the gain of feedback path is negative.

- If the value of (1+GH) is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, feedback will increase the sensitivity of the system gain in one frequency range and decrease in the other frequency range. Therefore,

we have to choose the values of 'GH' in such a way that the system is insensitive or less sensitive to parameter variations.

Effect of Feedback on Stability

- A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.

- In Equation 2, if the denominator value is zero (i.e., GH = -1), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore, we have to properly choose the feedback in order to make the control system stable.

Effect of Feedback on Noise

To know the effect of feedback on noise, let us compare the transfer function relations with and without feedback due to noise signal alone.

Consider an **open loop control system** with noise signal as shown below.
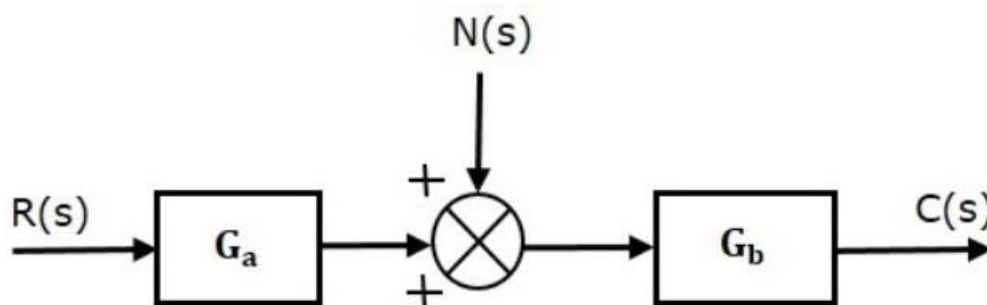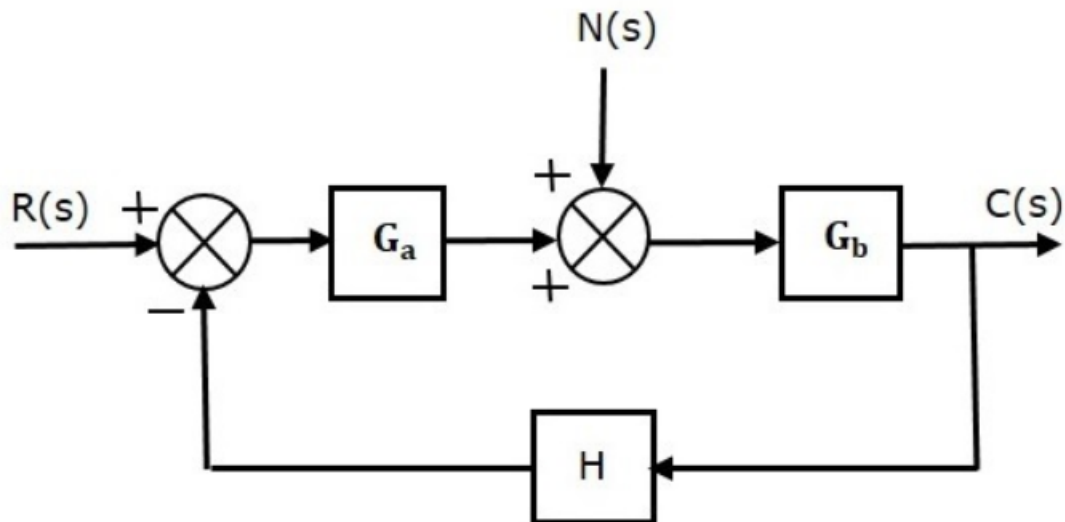

The **open loop transfer function** due to noise signal alone is

C(s)N(s)=Gb (Equation 7)

It is obtained by making the other input R(s) equal to zero.

Consider a **closed loop control system** with noise signal as shown below.

The **closed loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = \frac{G_b}{1 + G_a G_b H} \quad \text{(Equation 8)}$$

It is obtained by making the other input R(s) equal to zero.

Compare Equation 7 and Equation 8,

In the closed loop control system, the gain due to noise signal is decreased by a factor of $(1 + G_a G_b H)$ provided that the term $(1 + G_a G_b H)$ is greater than one.

Mathematical models – Differential equations

The set of mathematical equations describing the dynamic characteristics of a system is called 'Mathematical models' of the system. Mathematical models are used to convert mechanical to electrical or electrical to mechanical analogous equivalent circuits for easy analysis purpose.

These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

Transfer function of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.
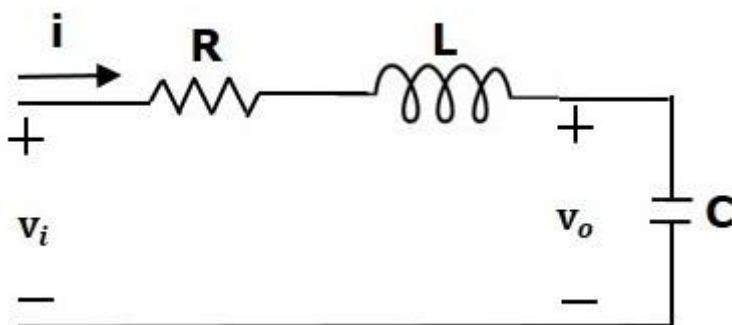
Differential Equation Model

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model.

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

Example

Consider the following electrical system as shown in the following figure. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in **series**. The input voltage applied to this circuit is vivi and the voltage across the capacitor is the output voltage vovo.

Mesh equation for this circuit is

$$v_i = Ri + L\frac{di}{dt} + v_o$$

Substitute, the current passing through capacitor in the above equation

$$i = c\frac{dv_o}{dt}$$

$$\Rightarrow v_i = RC\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2} + v_o$$

$$\Rightarrow \frac{d^2v_o}{dt^2} + \left(\frac{R}{L}\right)\frac{dv_o}{dt} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i$$

The above equation is a second order **differential equation**.

Modeling of Translational Mechanical Systems

Translational mechanical systems move along a **straight line**. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero. Let us now see the force opposed by these three elements individually.

Mass

Mass is the property of a body, which stores **kinetic energy**. If a force is applied on a body having mass **M**, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.
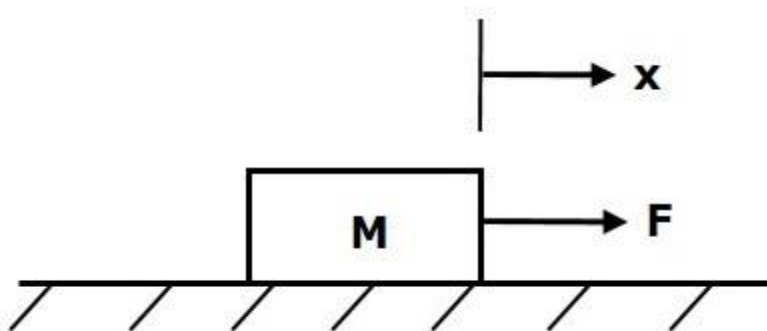
$$F_m \propto a$$

$$\Rightarrow F_m = Ma = M\frac{d^2 x}{dt^2}$$

$$F = F_m = M\frac{d^2 x}{dt^2}$$

Where,

- **F** is the applied force
- **Fm** is the opposing force due to mass
- **M** is mass
- **a** is acceleration
- **x** is displacement

Spring

Spring is an element, which stores **potential energy**. If a force is applied on spring **K**, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.
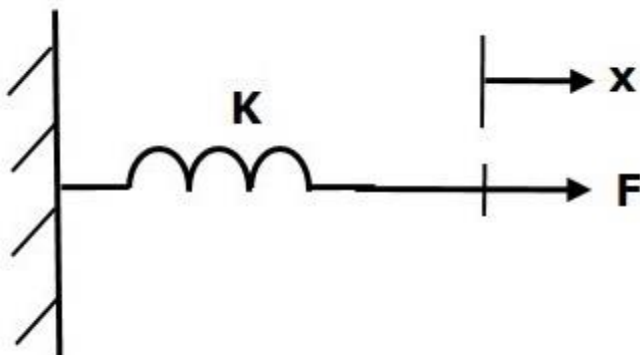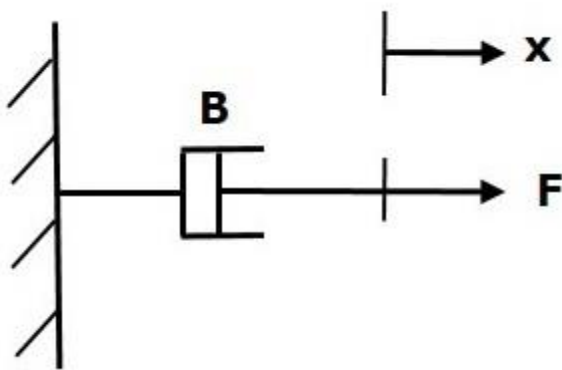

$$F \propto x$$

$$\Rightarrow F_k = Kx$$

$$F = F_k = Kx$$

Where,

- **F** is the applied force
- $\mathbf{F_k}$ is the opposing force due to elasticity of spring
- **K** is spring constant
- **x** is displacement

Dashpot

If a force is applied on dashpot **B**, then it is opposed by an opposing force due to **friction** of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible.

$$F_b \propto \nu$$

$$\Rightarrow F_b = B\nu = B\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$F = F_b = B\frac{\mathrm{d}x}{\mathrm{d}t}$$

Where,

- $\mathbf{F_b}$ is the opposing force due to friction of dashpot
- **B** is the frictional coefficient
- **v** is velocity
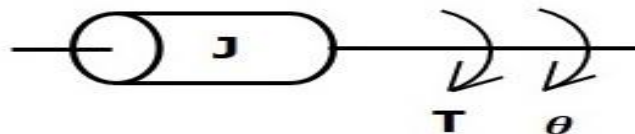- **x** is displacement

Modeling of Rotational Mechanical Systems

Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basic elements. Those are **moment of inertia, torsional spring** and **dashpot**.

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

Moment of Inertia

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores **kinetic energy**.

If a torque is applied on a body having moment of inertia **J**, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.

$$T_j \propto \alpha$$

$$\Rightarrow T_j = J\alpha = J\frac{d^2\theta}{dt^2}$$

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

Where,

- **T** is the applied torque
- **T$_j$** is the opposing torque due to moment of inertia
- **J** is moment of inertia
- **α** is angular acceleration

- **θ** is angular displacement

Torsional Spring

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores **potential energy**.

If a torque is applied on torsional spring **K**, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.
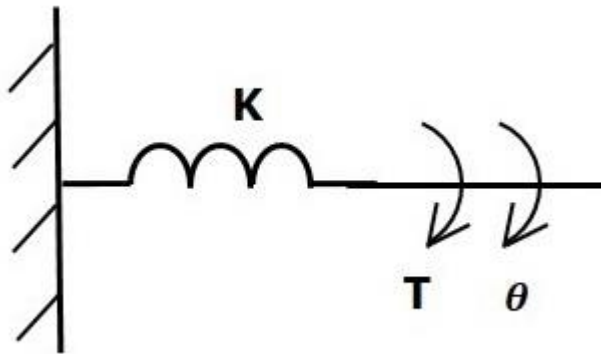

$$T_k \propto \theta$$

$$\Rightarrow T_k = K\theta$$

$$T = T_k = K\theta$$

Where,

- **T** is the applied torque
- **T_k** is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- **θ** is angular displacement

Dashpot

If a torque is applied on dashpot **B**, then it is opposed by an opposing torque due to the **rotational friction** of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.
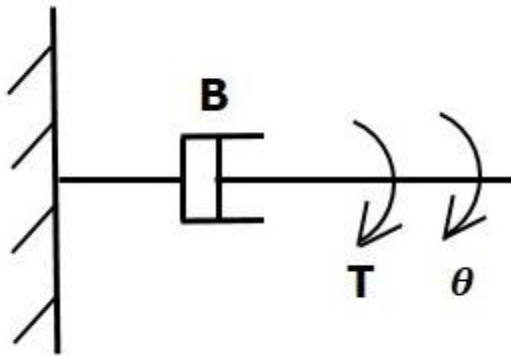
$$T_b \propto \omega$$

$$\Rightarrow T_b = B\omega = B\frac{d\theta}{dt}$$

$$T = T_b = B\frac{d\theta}{dt}$$

Where,

- **T$_b$** is the opposing torque due to the rotational friction of the dashpot
- **B** is the rotational friction coefficient
- ω is the angular velocity
- θ is the angular displacement

Two systems are said to be **analogous** to each other if the following two conditions are satisfied.

- The two systems are physically different
- Differential equation modelling of these two systems are same

Electrical systems and mechanical systems are two physically different systems. There are two types of electrical analogies of translational mechanical systems. Those are force voltage analogy and force current analogy.

Force Voltage Analogy

In force voltage analogy, the mathematical equations of **translational mechanical system** are compared with mesh equations of the electrical system.

Consider the following translational mechanical system as shown in the following figure.
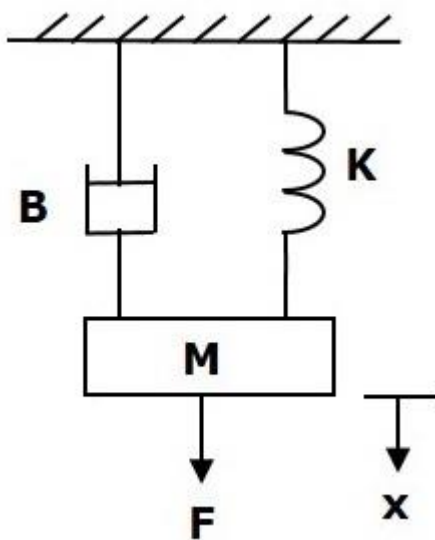
The **force balanced equation** for this system is

$$F = F_m + F_b + F_k$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad \textbf{(Equation 1)}$$

Consider the following electrical system as shown in the following figure. This circuit consists of a resistor, an inductor and a capacitor. All these electrical elements are connected in a series. The input voltage applied to this circuit is VV volts and the current flowing through the circuit is ii Amps.
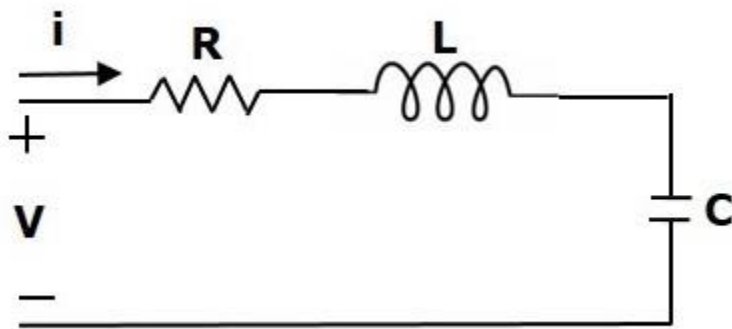
Mesh equation for this circuit is

$$V = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt \qquad \text{(Equation 2)}$$

Substitute, $i = \frac{dq}{dt}$ in Equation 2.

$$V = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

$$\Rightarrow V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \left(\frac{1}{c}\right)q \qquad \text{(Equation 3)}$$

By comparing Equation 1 and Equation 3, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Translational Mechanical System	Electrical System
Force(F)	Voltage(V)
Mass(M)	Inductance(L)
Frictional Coefficient(B)	Resistance(R)

Spring Constant(K)	Reciprocal of Capacitance (1c)(1c)
Displacement(x)	Charge(q)
Velocity(v)	Current(i)

Similarly, there is torque voltage analogy for rotational mechanical systems. Let us now discuss about this analogy.

Torque Voltage Analogy

In this analogy, the mathematical equations of **rotational mechanical system** are compared with mesh equations of the electrical system.

Rotational mechanical system is shown in the following figure.
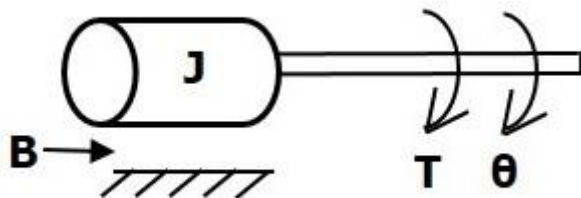
The torque balanced equation is

$$T = T_j + T_b + T_k$$

$$\Rightarrow T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + k\theta \qquad \textbf{(Equation 4)}$$

By comparing Equation 4 and Equation 3, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.
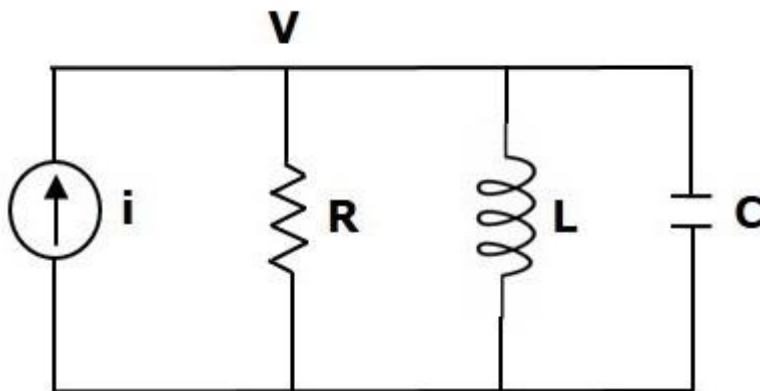
Rotational Mechanical System	Electrical System
Torque(T)	Voltage(V)
Moment of Inertia(J)	Inductance(L)

Rotational friction coefficient(B)	Resistance(R)
Torsional spring constant(K)	Reciprocal of Capacitance (1c)(1c)
Angular Displacement(θ)	Charge(q)
Angular Velocity(ω)	Current(i)

Force Current Analogy

In force current analogy, the mathematical equations of the **translational mechanical system** are compared with the nodal equations of the electrical system.

Consider the following electrical system as shown in the following figure. This circuit consists of current source, resistor, inductor and capacitor. All these electrical elements are connected in parallel.

The nodal equation is

$$i = \frac{V}{R} + \frac{1}{L}\int V dt + C\frac{dV}{dt} \qquad \text{(Equation 5)}$$

Substitute, $V = \frac{d\Psi}{dt}$ in Equation 5.

$$i = \frac{1}{R}\frac{d\Psi}{dt} + \left(\frac{1}{L}\right)\Psi + C\frac{d^2\Psi}{dt^2}$$

$$\Rightarrow i = C\frac{d^2\Psi}{dt^2} + \left(\frac{1}{R}\right)\frac{d\Psi}{dt} + \left(\frac{1}{L}\right)\Psi \qquad \text{(Equation 6)}$$

By comparing Equation 1 and Equation 6, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Translational Mechanical System	Electrical System
Force(F)	Current(i)
Mass(M)	Capacitance(C)
Frictional coefficient(B)	Reciprocal of Resistance(1R)(1R)
Spring constant(K)	Reciprocal of Inductance(1L)(1L)
Displacement(x)	Magnetic Flux(ψ)
Velocity(v)	Voltage(V)

Similarly, there is a torque current analogy for rotational mechanical systems. Let us now discuss this analogy.

Torque Current Analogy

In this analogy, the mathematical equations of the **rotational mechanical system** are compared with the nodal mesh equations of the electrical system.

By comparing Equation 4 and Equation 6, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.

Rotational Mechanical System	Electrical System
Torque(T)	Current(i)
Moment of inertia(J)	Capacitance(C)

Rotational friction coefficient(B)	Reciprocal of Resistance(1R)(1R)
Torsional spring constant(K)	Reciprocal of Inductance(1L)(1L)
Angular displacement(θ)	Magnetic flux(ψ)
Angular velocity(ω)	Voltage(V)

These analogies are helpful to study and analyze the non-electrical system like mechanical system from analogous electrical system.

Impulse Response

In control theory the impulse response is the response of a system to a Dirac delta input. This proves useful in the analysis of dynamic systems; the Laplace transform of the delta function is 1, so the impulse response is equivalent to the inverse Laplace transform of the system's transfer function.

An impulse response means the output/behaviour of a system/process when we provide it with an impulse signal. An impulse signal is a momentary signal of infinte magnitude (ideally).

The Laplace transform of an impulse function is 1, so the impulse response is equivalent to the inverse Laplace transform of the system's transfer function, for an LTI system. In other words, for an unknown system, its impulse response can help us understand the system as for any input to a system, the output can be calculated in terms of the input and the impulse response. This is possible because we can decompose our input signal into a sum of scaled and time-shifted impulses and following the properties of an LTI system, the output would be equal to the sum of the impulse response, scaled and time-shifted in the same way.

a unit impulse or Dirac's function is a function of time which satisfies,

- $\delta(t) = 0$ for all $t \neq 0$ and
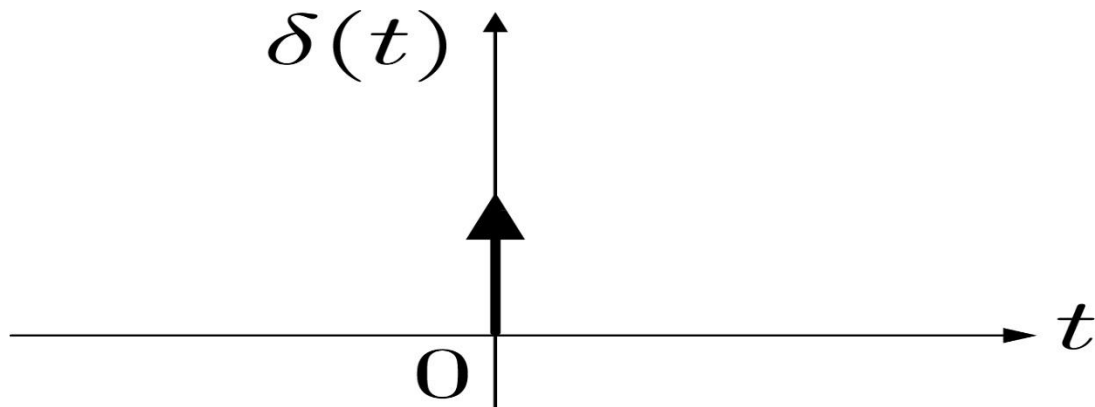- $\int_{-a}^{a} \delta(t)dt = 1$ for all $a > 0$.

Figure: Unit impulse or Dirac's function

Transfer functions

A **transfer function** represents the relationship between the output signal of a control system and the input signal, for all possible input values. A block diagram is a visualization of the control system which uses blocks to represent the transfer function, and arrows which represent the various input and output signals.

For any control system, there exists a reference input known as excitation or cause which operates through a transfer operation (i.e. the transfer function) to produce an effect resulting in controlled output or response.

Thus the cause and effect relationship between the output and input is related to each other through a **transfer function**.

In a Laplace Transform, if the input is represented by R(s) and the output is represented by C(s), then the transfer function will be:

$$G(s) = \frac{C(s)}{R(s)} \Rightarrow R(s).G(s) = C(s)$$

That is, the transfer function of the system multiplied by the input function gives the output function of the system.

The transfer function of a control system is defined as the ratio of the Laplace transform of the output variable to Laplace transform of the input variable assuming all initial conditions to be zero.

$$G(s) = \frac{C(s)}{R(s)}$$

Procedure for determining the transfer function of a control system are as follows:

1. We form the equations for the system.
2. Now we take Laplace transform of the system equations, assuming initial conditions as zero.
3. Specify system output and input.
4. Lastly we take the ratio of the Laplace transform of the output and the Laplace transform of the input which is the required transfer function.

It is not necessary that output and input of a control system are of same category. For example, in electric motors the input is electrical signal whereas the output is mechanical signal since electrical energy required to rotate the motors. Similarly in an electric generator, the input is mechanical signal and the output is electrical signal, since mechanical energy is required to produce electricity in a generator.

But for mathematical analysis, of a system all kinds of signals should be represented in a similar form. This is done by transforming all kinds of signal to their Laplace form. Also the transfer function of a system is represented by Laplace form by dividing output Laplace transfer function to input Laplace transfer function. Hence a basic block diagram of a control system can be represented as

$$Where,\ R(s) = £r(t),\ C(s) = £c(t)\ \&\ G(s) = \frac{£c(t)}{£r(t)}$$

Where r(t) and c(t) are time domain function of input and output signal respectively.

TRANSFER FUNCTION REPRESENTATION

Block diagram representation of systems considering electrical systems as examples

A control system can be represented by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted in to algebraic equations.

From the working knowledge of the system the input and output variables are identified. Each equation gives one section of block diagram. The output of one section will be input of another section.

Obtain the current flowing through series elements and voltage across shunt elements. Various sections are interconnected to obtain the overall block diagram of the system.

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.
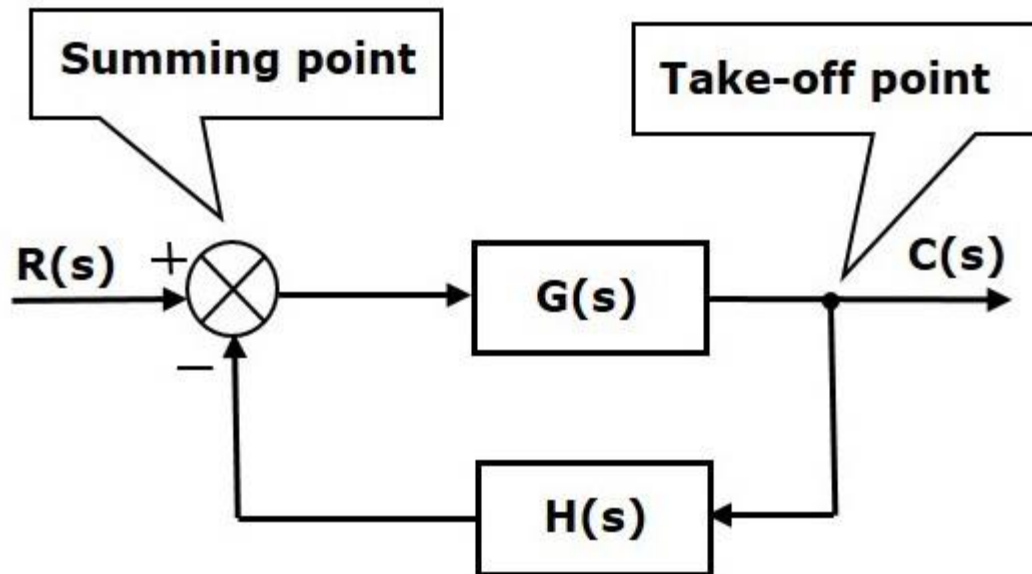
The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

Transfer Function,

$$G(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow Y(s) = G(s)X(s)$$

Output of the block is obtained by multiplying transfer function of the block with input.

Summing Point

The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.
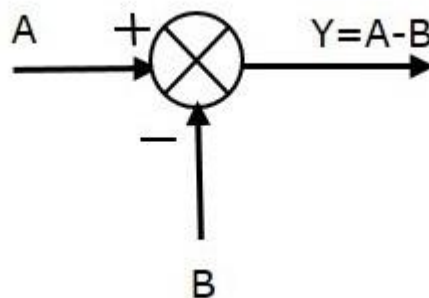
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B.**

i.e., Y = A + B.
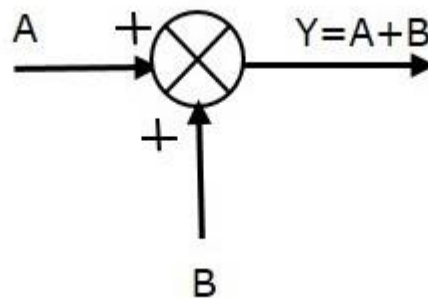
The following figure shows the summing point with two inputs (A, B) and one output (Y).

Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B**.

Y = A + (-B) = A - B.

The following figure shows the summing point with three inputs (A, B, C) and one output (Y).

Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as
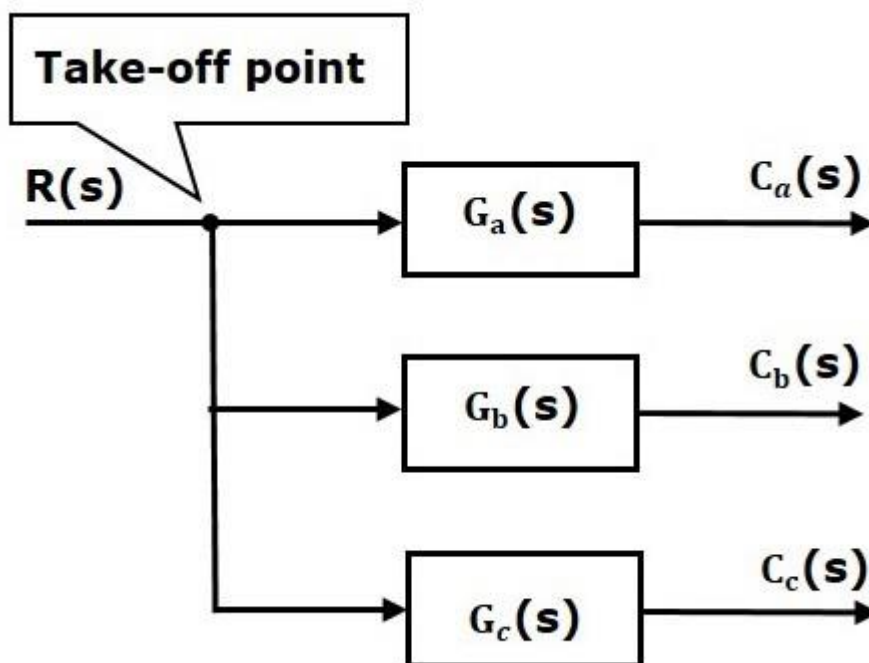
Y = A + B + (−C) = A + B − C.
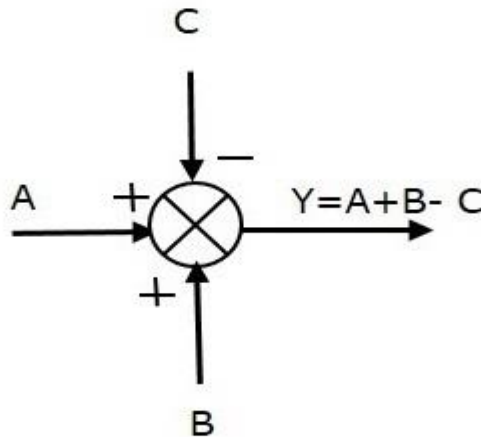
Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points.

In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.

In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.
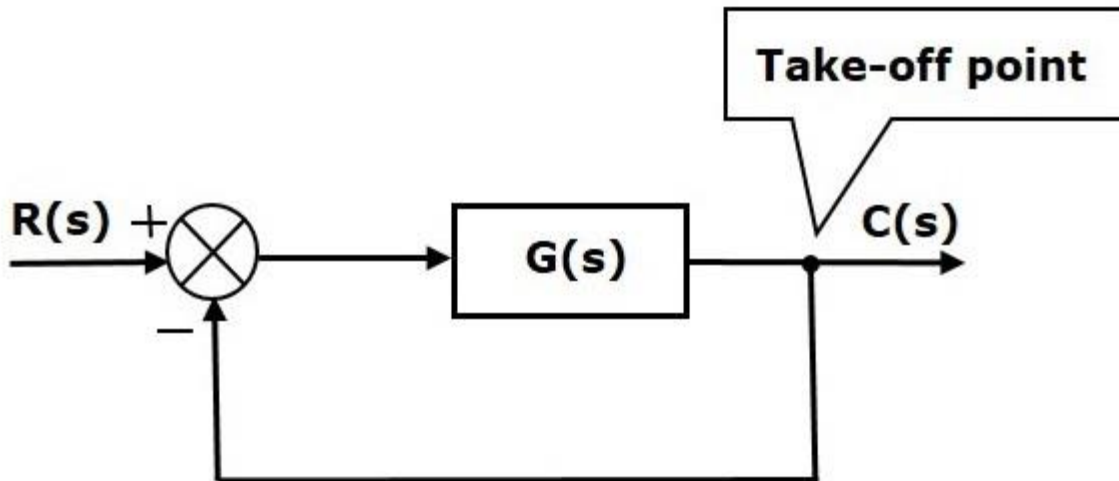
Electrical systems contain mainly three basic elements — **resistor, inductor and capacitor**.

Consider a series of RLC circuit as shown in the following figure. Where, $V_i(t)$ and $V_o(t)$ are the input and output voltages. Let $i(t)$ be the current passing through the circuit. This circuit is in time domain.
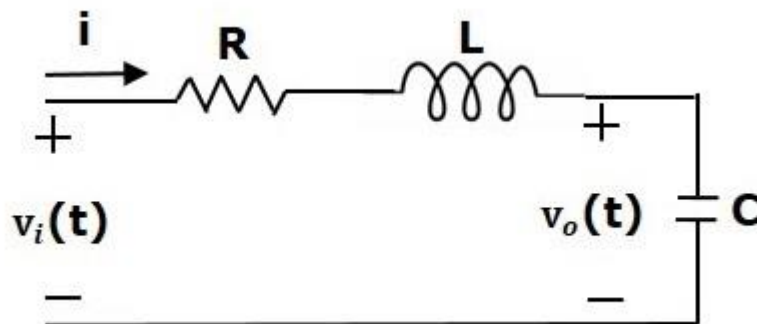
By applying the Laplace transform to this circuit, will get the circuit in s-domain. The circuit is as shown in the following figure.
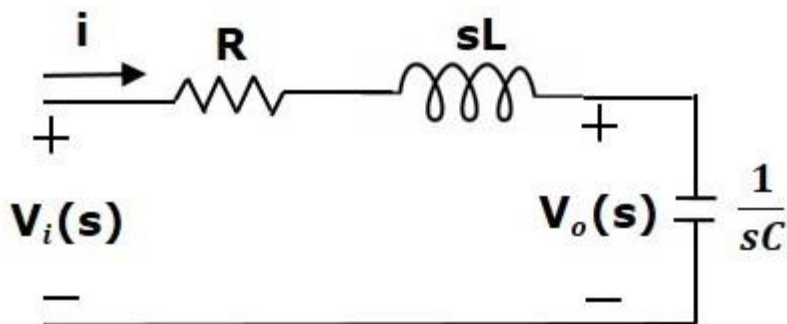

From the above circuit, we can write

$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$

$$\Rightarrow I(s) = \left\{ \frac{1}{R+sL} \right\} \{V_i(s) - V_o(s)\} \qquad \textbf{(Equation 1)}$$

$$V_o(s) = \left(\frac{1}{sC} \right) I(s) \qquad \textbf{(Equation 2)}$$

Let us now draw the block diagrams for these two equations individually. And then combine those block diagrams properly in order to get the overall block diagram of series of RLC Circuit (s-domain).

Equation 1 can be implemented with a block having the transfer function, 1R+sL1R+sL. The input and output of this block are {Vi(s)−Vo(s)} {Vi(s)−Vo(s)} and I(s)I(s). We require a summing point to get {Vi(s)−Vo(s)} {Vi(s)−Vo(s)}. The block diagram of Equation 1 is shown in the following figure.
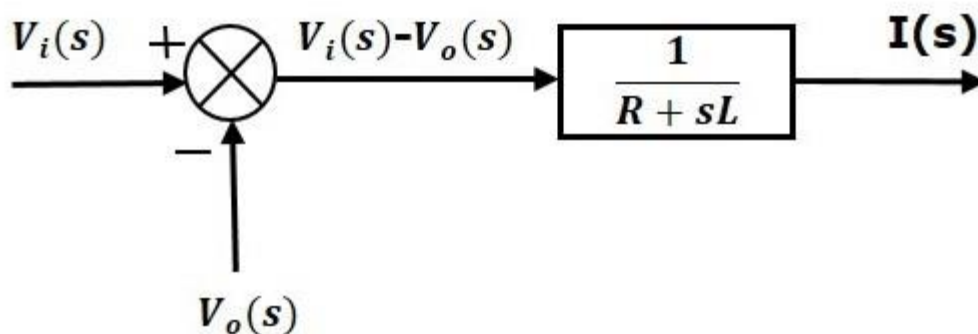
Equation 2 can be implemented with a block having transfer function, 1sC1sC. The input and output of this block are I(s)I(s) and Vo(s)Vo(s). The block diagram of Equation 2 is shown in the following figure.
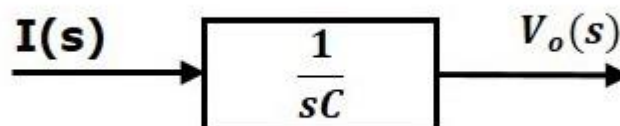
The overall block diagram of the series of RLC Circuit (s-domain) is shown in the following figure.
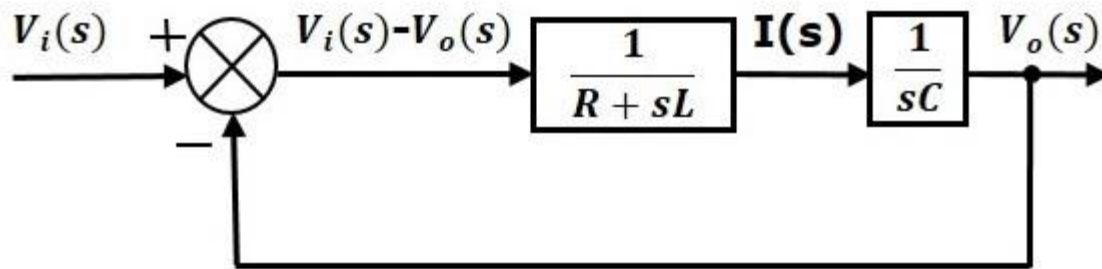
Similarly, you can draw the block diagram of any electrical circuit or system just by following this simple procedure.

- Convert the time domain electrical circuit into an s-domain electrical circuit by applying Laplace transform.
- Write down the equations for the current passing through all series branch elements and voltage across all shunt branches.
- Draw the block diagrams for all the above equations individually.
- Combine all these block diagrams properly in order to get the overall block diagram of the electrical circuit (s-domain).

Block diagram algebra

Block diagram algebra is nothing but the algebra involved with the basic elements of the block diagram. This algebra deals with the pictorial representation of algebraic equations.

Basic Connections for Blocks

There are three basic types of connections between two blocks.

Series Connection

Series connection is also called **cascade connection**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in series.
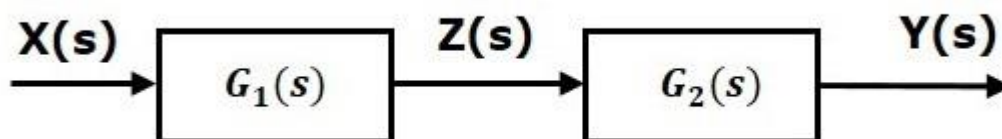
For this combination, we will get the output Y(s) as

$$Y(s) = G_2(s)Z(s)$$

Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$
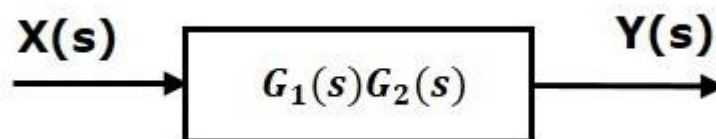
$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare   this   equation   with   the   standard   form   of   the   output
equation, Y(s)=G(s)X(s)Y(s)=G(s)X(s). Where, G(s)=G1(s)G2(s)G(s)=G1(s)G2(s).

That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.

Similarly, you can represent series connection of 'n' blocks with a single block. The transfer function of this single block is the product of the transfer functions of all those 'n' blocks.

Parallel Connection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in parallel. The outputs of these two blocks are connected to the summing point.

For this combination, we will get the output Y(s) as

$$Y(s) = Y_1(s) + Y_2(s)$$

Where

$$Y_1(s) = G_1(s)X(s) \quad \text{and} \quad Y_2(s) = G_2(s)X(s)$$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, Y(s)=G(s)X(s)
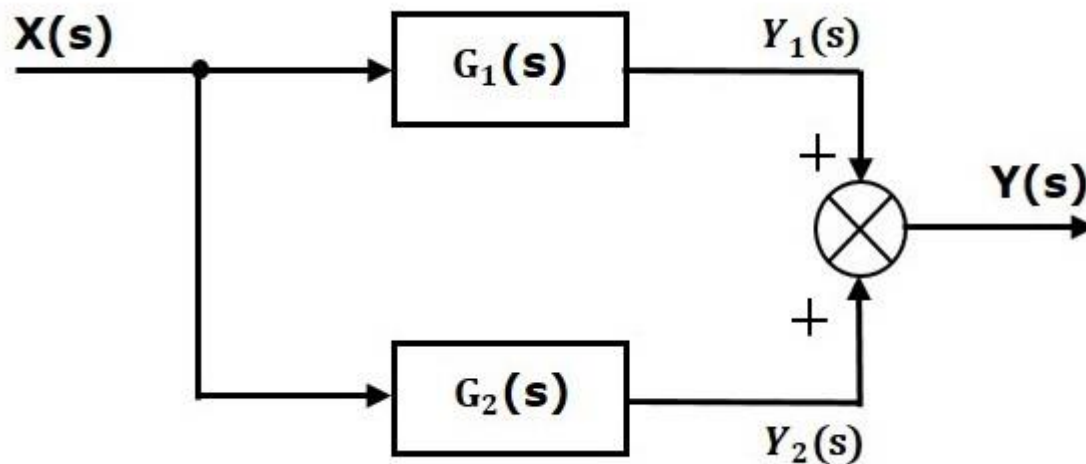Where, G(s)=G1(s)+G2(s).

That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of those two blocks. The equivalent block diagram is shown below.
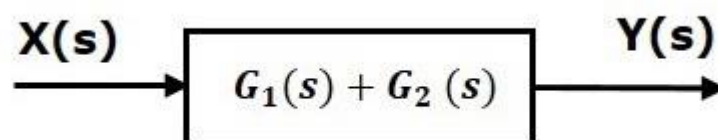
Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

Feedback Connection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions G(s)G(s) and H(s)H(s) form a closed loop.
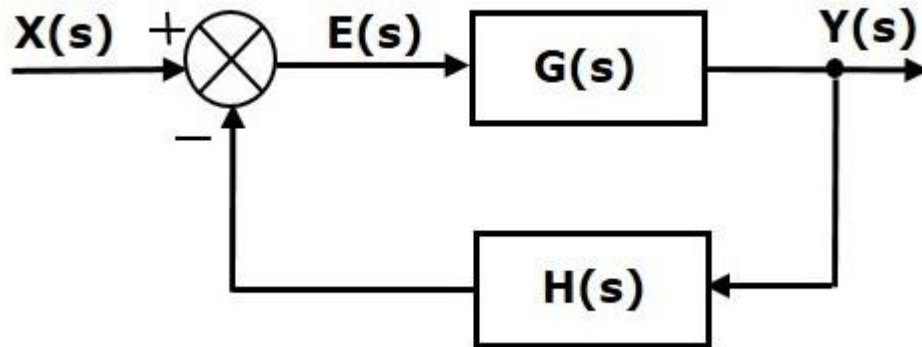
The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output Y(s)Y(s) is –

$$Y(s) = E(s)G(s)$$

Substitute E(s)E(s) value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore, the negative feedback closed loop transfer function is

$$\frac{G(s)}{1+G(s)H(s)}$$

Similarly, we can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e.,

$$\frac{G(s)}{1-G(s)H(s)}$$

Block Diagram Algebra for Summing Points

There are two possibilities of shifting summing points with respect to blocks −

- Shifting summing point after the block
- Shifting summing point before the block

Let us now see what kind of arrangements need to be done in the above two cases one by one.

Shifting Summing Point After the Block

Consider the block diagram shown in the following figure. Here, the summing point is present before the block.
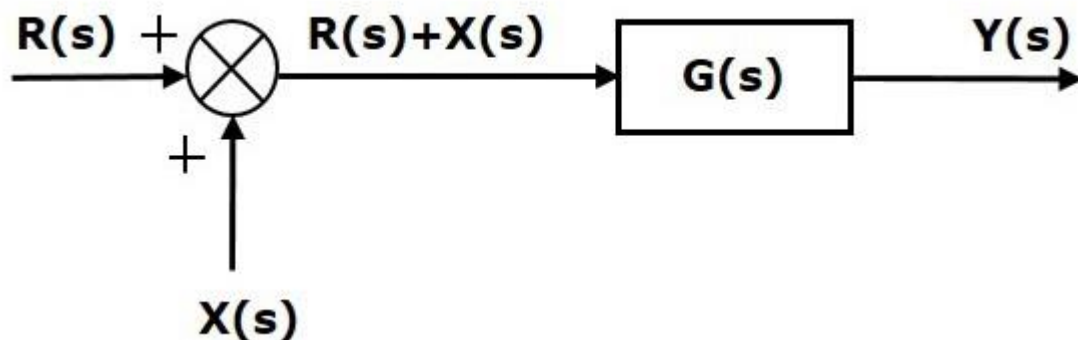
Summing point has two inputs R(s) and X(s). The output of it is {R(s)+X(s).

So, the input to the block G(s)is {R(s)+X(s)} and the output of it is −

Y(s)=G(s){R(s)+X(s)}

⇒Y(s)=G(s)R(s)+G(s)X(s)⇒Y(s) (Equation 1)

Now, shift the summing point after the block. This block diagram is shown in the following figure.

Output of the block G(s) is G(s)R(s).The output of the summing point is

Y(s)=G(s)R(s)+X(s) **(Equation 2)** Compare Equation 1 and Equation 2.

The first term 'G(s)R(s)' is same in both the equations. But there is difference in the second term. In order to get the second term also same, we require one more block G(s). It is having the input X(s) and the output of this block is given as input to summing point instead of X(s). This block diagram is shown in the following figure.
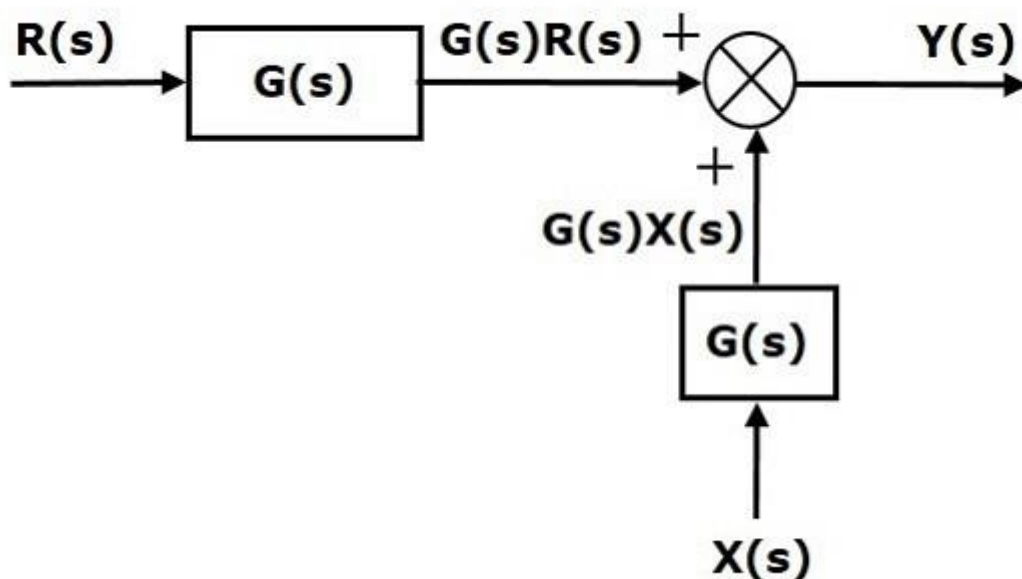
Shifting Summing Point Before the Block

Consider the block diagram shown in the following figure. Here, the summing point is present after the block.
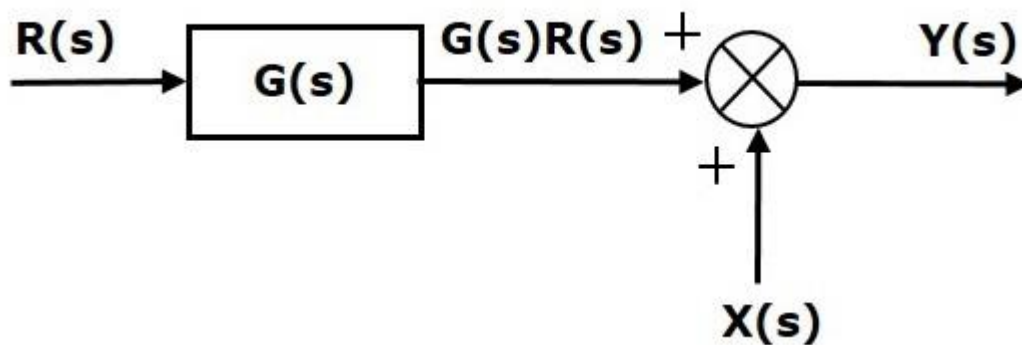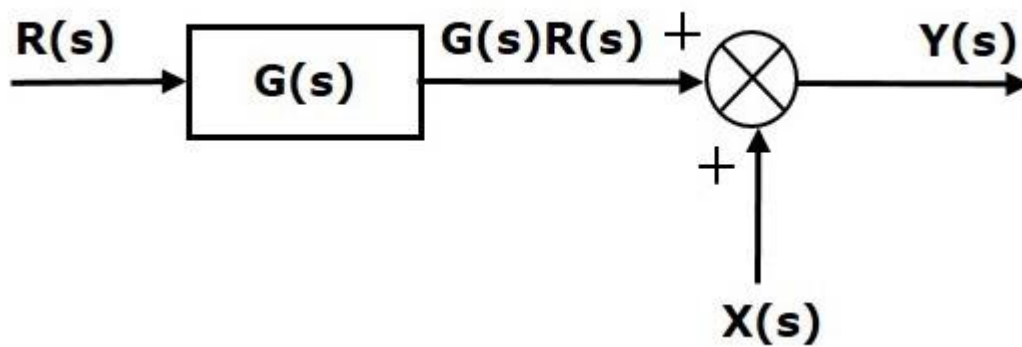
Output of this block diagram is -

Y(s)=G(s)R(s)+X(s)Y(s) **(Equation 3)**

Now, shift the summing point before the block. This block diagram is shown in the following figure.
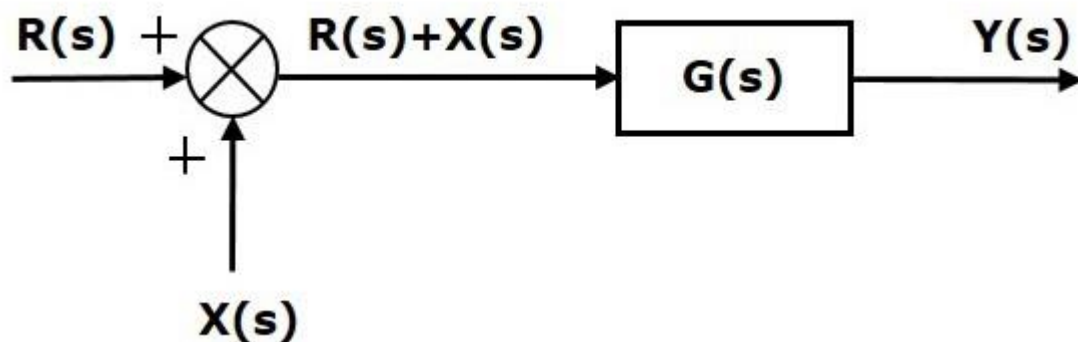
Output of this block diagram is -Y(S)=G(s)R(s)+G(s)X(s)Y(S)   **(Equation 4)**

Compare Equation 3 and Equation 4, The first term 'G(s)R(s)'is same in both equations. But, there is difference in the second term. In order to get the second term also same, we require one more block 1G(s). It is having the input X(s) and the output of this block is given as input to summing point instead of X(s). This block diagram is shown in the following figure.
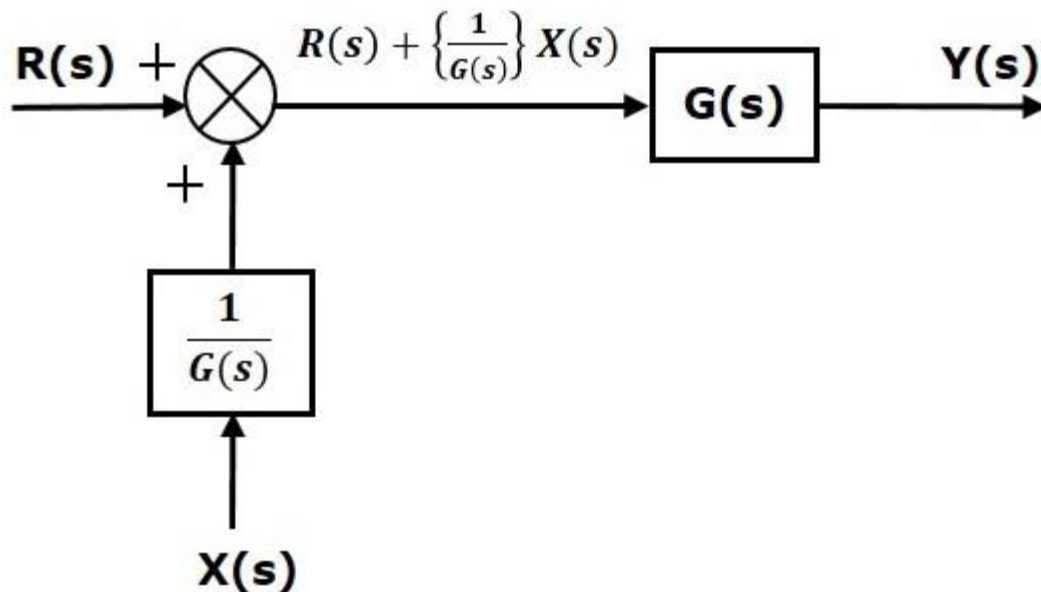
Block Diagram Algebra for Take-off Points

There are two possibilities of shifting the take-off points with respect to blocks −

- Shifting take-off point after the block

- Shifting take-off point before the block

Let us now see what kind of arrangements are to be done in the above two cases, one by one.

Shifting Take-off Point After the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.
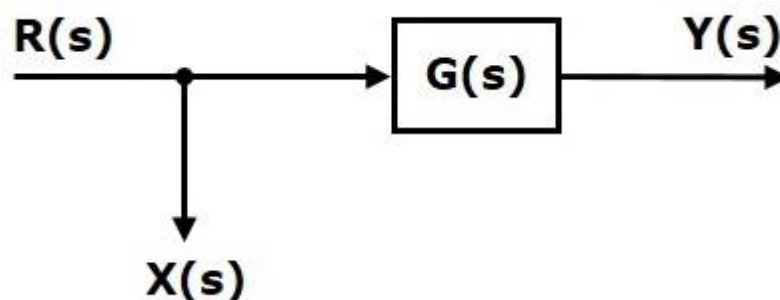
Here, $X(s)=R(s)$ and $Y(s)=G(s)R(s)$

When you shift the take-off point after the block, the output $Y(s)$ will be same. But, there is difference in $X(s)$ value. So, in order to get the same $X(s)$ value, we require one more

block $\frac{1}{G(s)}$. It is having the input Y(s) and the output is X(s). This block diagram is shown in the following figure.
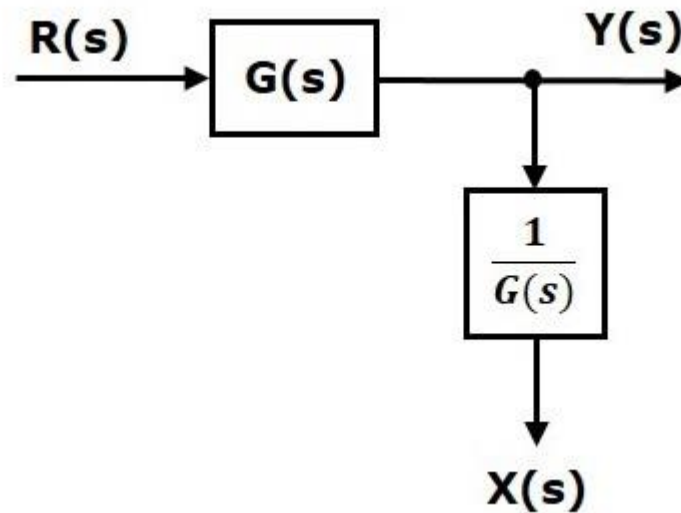
R(s) →[G(s)]→ Y(s)

$\dfrac{1}{G(s)}$

X(s)

Shifting Take-off Point Before the Block

Consider the block diagram shown in the following figure. Here, the take-off point is present after the block.

R(s) →[G(s)]→ Y(s)

X(s)
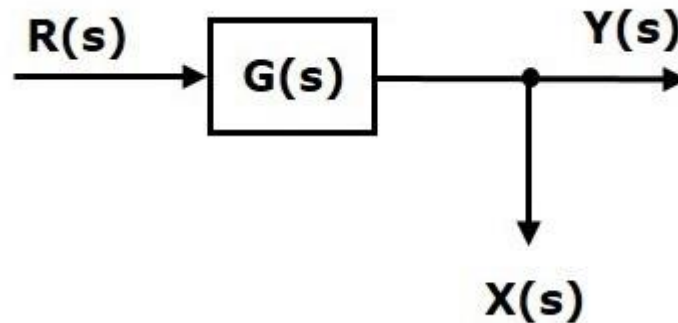
Here, $X(s)=Y(s)=G(s)R(s)$ When you shift the take-off point before the block, the output Y(s) will be same. But there is difference in X(s) value. So, in order to get same X(s) value, we require one more block G(s). It is having the input R(s) and the output is X(s). This block diagram is shown in the following figure.
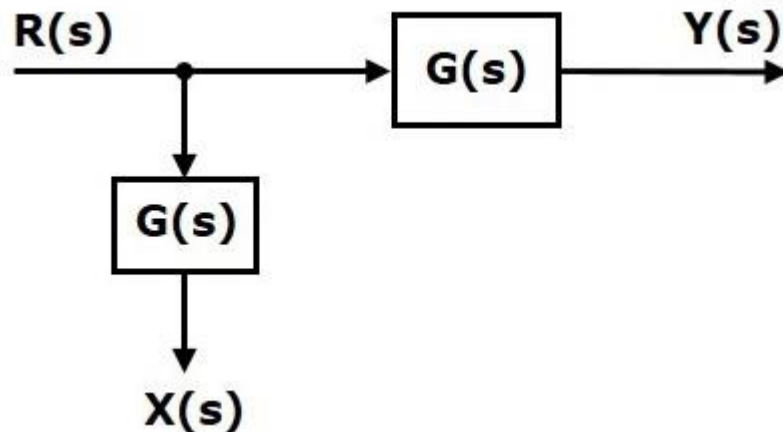
Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

- **Rule 1** − Check for the blocks connected in series and simplify.

- **Rule 2** − Check for the blocks connected in parallel and simplify.

- **Rule 3** − Check for the blocks connected in feedback loop and simplify.

- **Rule 4** − If there is difficulty with take-off point while simplifying, shift it towards right.

- **Rule 5** − If there is difficulty with summing point while simplifying, shift it towards left.

- **Rule 6** − Repeat the above steps till you get the simplified form, i.e., single block.

Note − The transfer function present in this single block is the transfer function of the overall block diagram.

Example

Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.
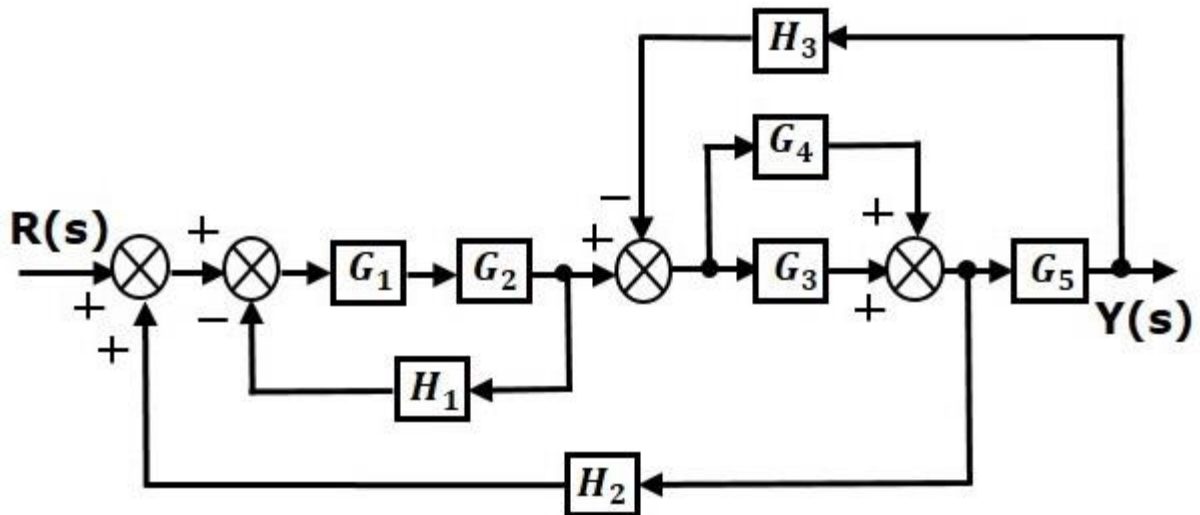
Step 1 − Use Rule 1 for blocks G1 and G2. Use Rule 2 for blocks G3 and G4. The modified block diagram is shown in the following figure.
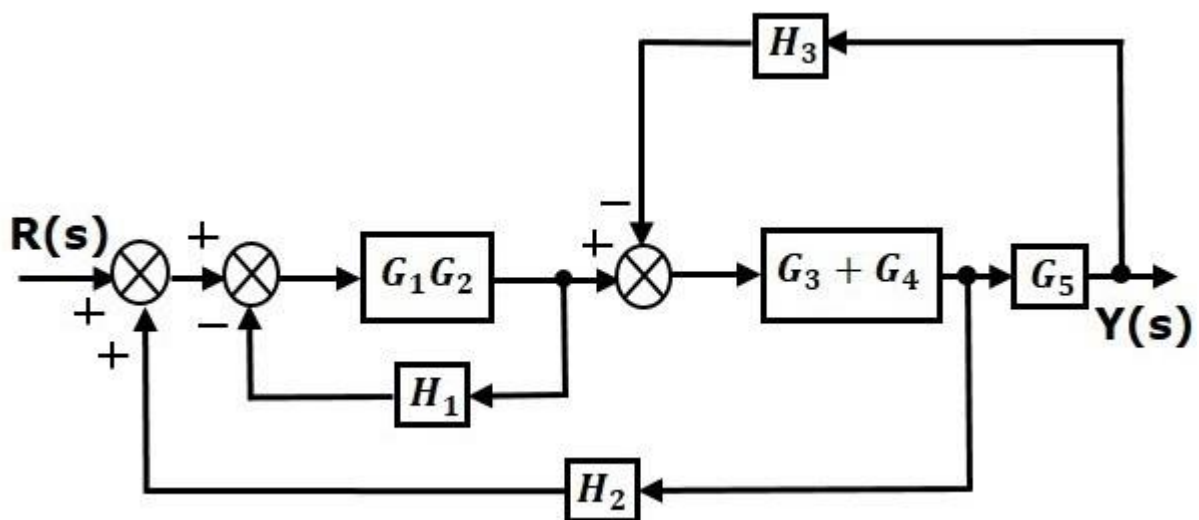
Step 2 − Use Rule 3 for blocks G1G2 and H1. Use Rule 4 for shifting take-off point after the block G5. The modified block diagram is shown in the following figure.
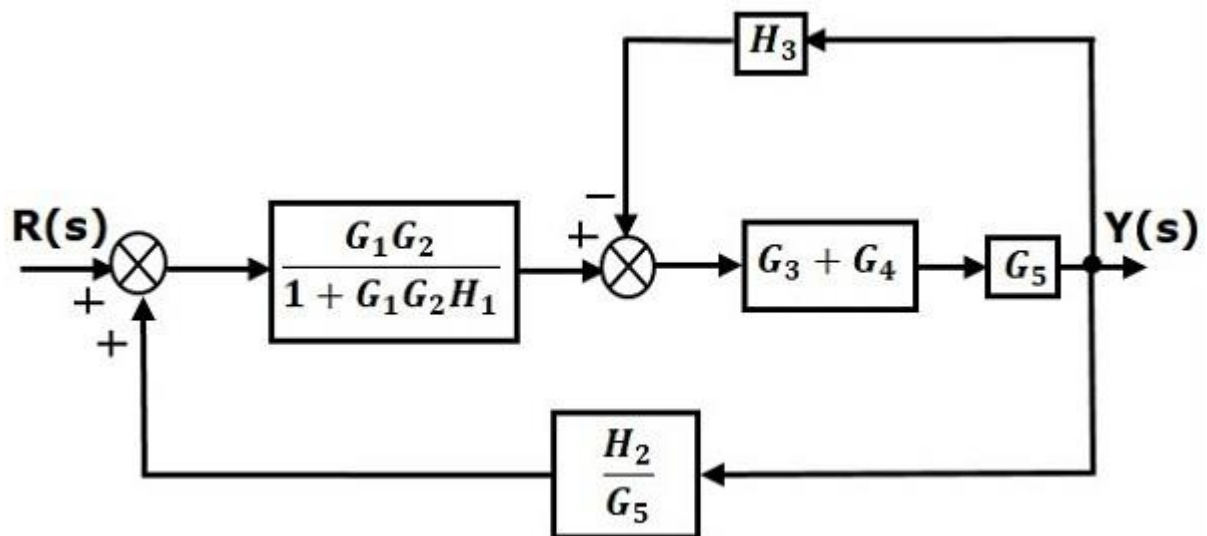
Step 3 − Use Rule 1 for blocks (G3+G4) and G5. The modified block diagram is shown in the following figure.
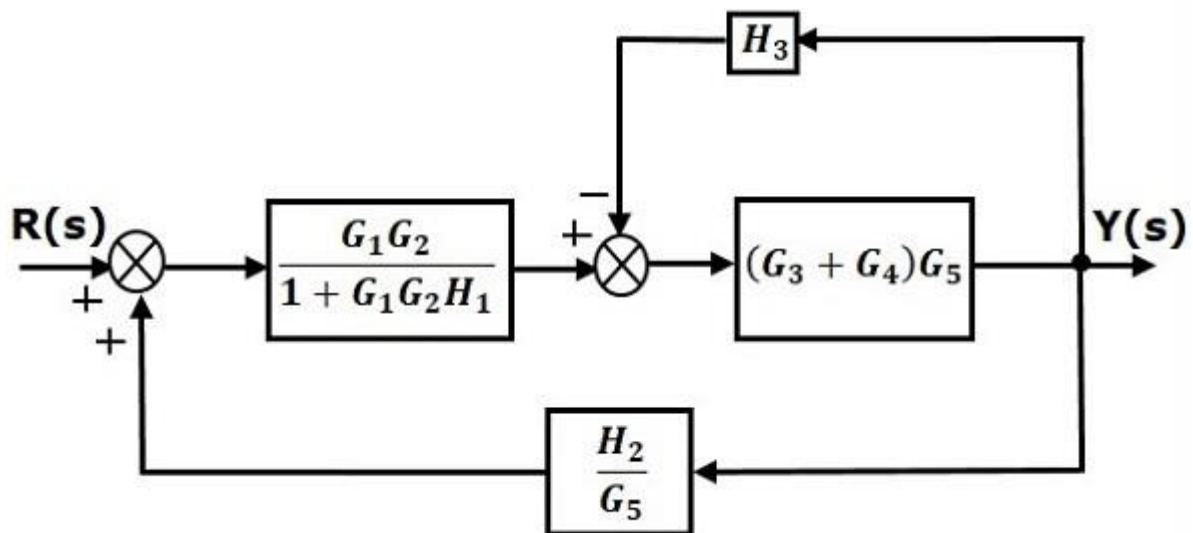
Step 4 − Use Rule 3 for blocks (G3+G4)G5 and H3. The modified block diagram is shown in the following figure.
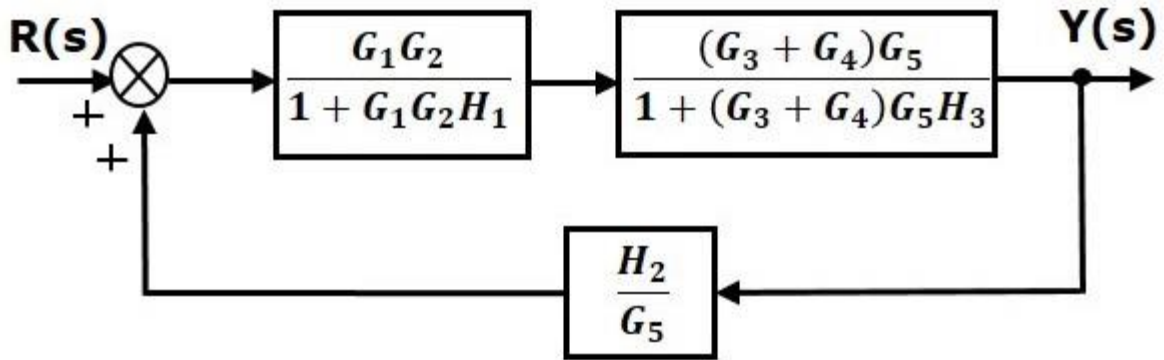
Step 5 − Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.
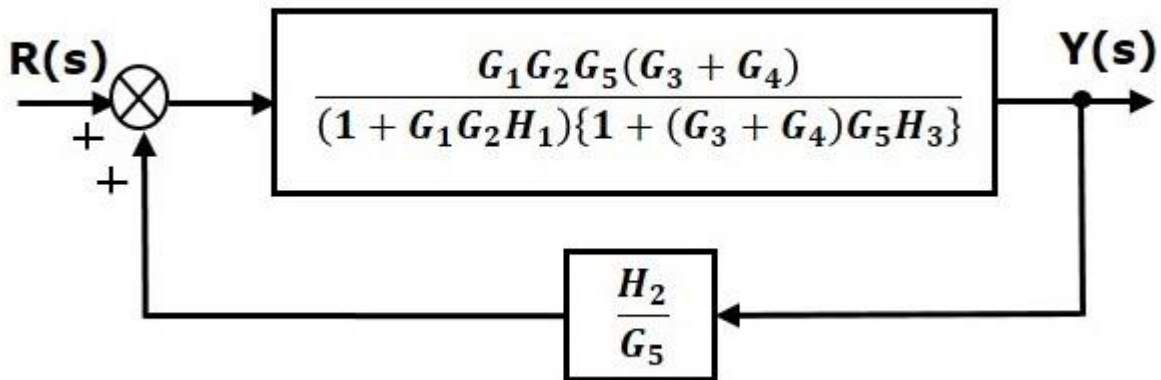
Step 6 − Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.
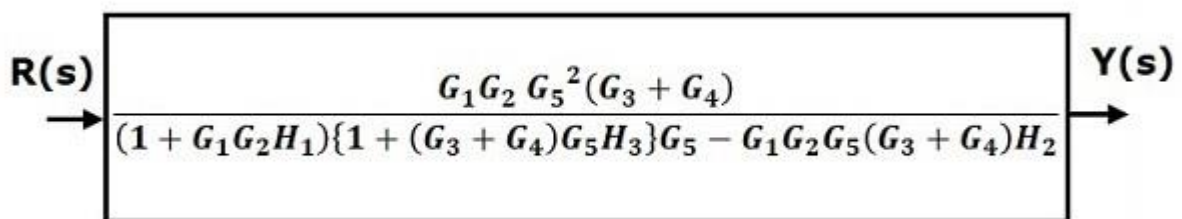
Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_5^2 (G_3 + G_4)}{(1 + G_1 G_2 H_1)\{1 + (G_3 + G_4)G_5 H_3\}G_5 - G_1 G_2 G_5 (G_3 + G_4)H_2}$$

Note − Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

- **Step 1** − Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.

- **Step 2** − Repeat step 1 for remaining inputs.

- **Step 3** − Get the overall transfer function by adding all those transfer functions.

Representation by Signal flow graph

The block diagram reduction process takes more time for complicated systems. Because, we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

Basic Elements of Signal Flow Graph

Nodes and branches are the basic elements of signal flow graph.

Node

Node is a point which represents either a variable or a signal. There are three types of nodes — input node, output node and mixed node.

Input Node − It is a node, which has only outgoing branches.

Output Node − It is a node, which has only incoming branches.

Mixed Node − It is a node, which has both incoming and outgoing branches.

Example

Let us consider the following signal flow graph to identify these nodes.
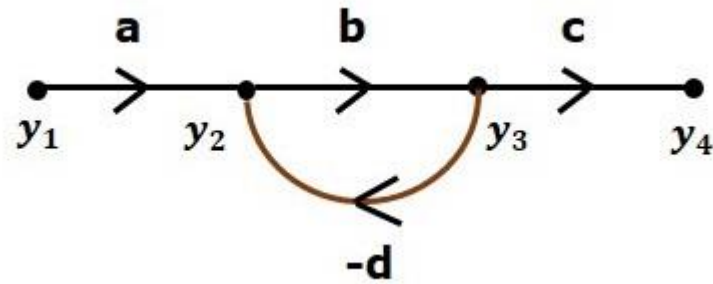
The nodes present in this signal flow graph are y1, y2, y3 and y4.

y1 and y4 are the input node and output node respectively.

y2 and y3 are mixed nodes.

Branch

Branch is a line segment which joins two nodes. It has both gain and direction. For example, there are four branches in the above signal flow graph. These branches have gains of a, b, c and -d.

Construction of Signal Flow Graph

Let us construct a signal flow graph by considering the following algebraic equations −

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six nodes (y1, y2, y3, y4, y5 and y6) and eight branches in this signal flow graph. The gains of the branches are a12, a23, a34, a45, a56, a42, a53 and a35.

To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below −

Step 1 − Signal flow graph for y2=a13y1+a42y4 is shown in the following figure.
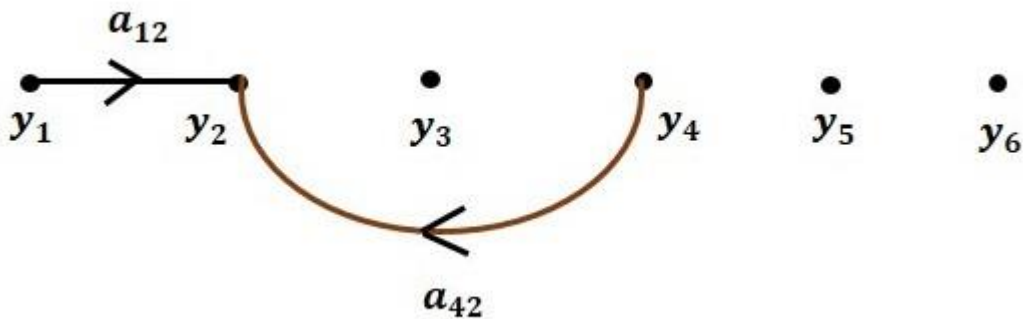
Step 2 − Signal flow graph for y3=a23y2+a53y5 is shown in the following figure.
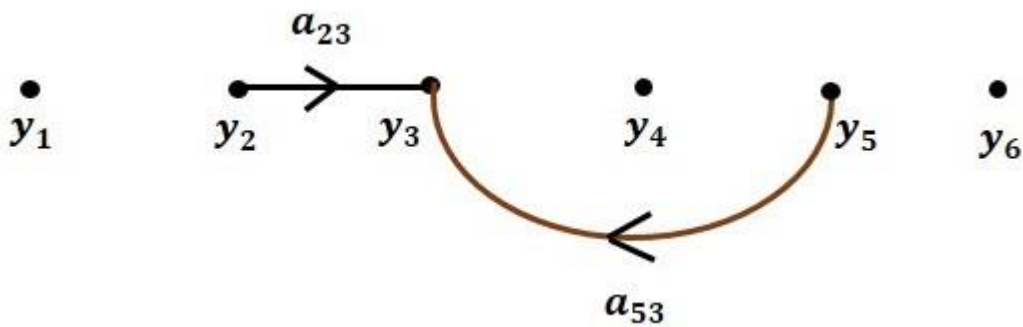
Step 3 − Signal flow graph for y4=a34y3 is shown in the following figure.
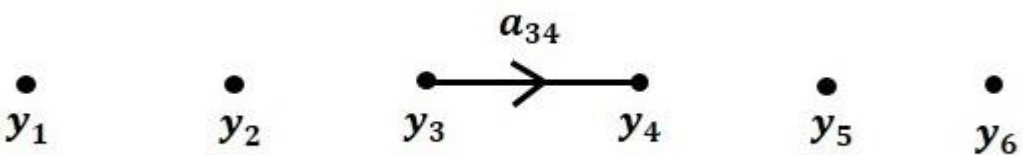
Step 4 − Signal flow graph for y5=a45y4+a35y3 is shown in the following figure.
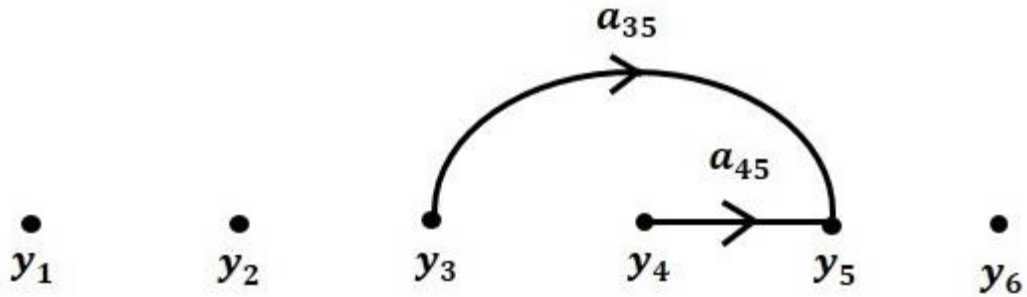
Step 5 − Signal flow graph for y6=a56y5 is shown in the following figure.

Step 6 − Signal flow graph of overall system is shown in the following figure.
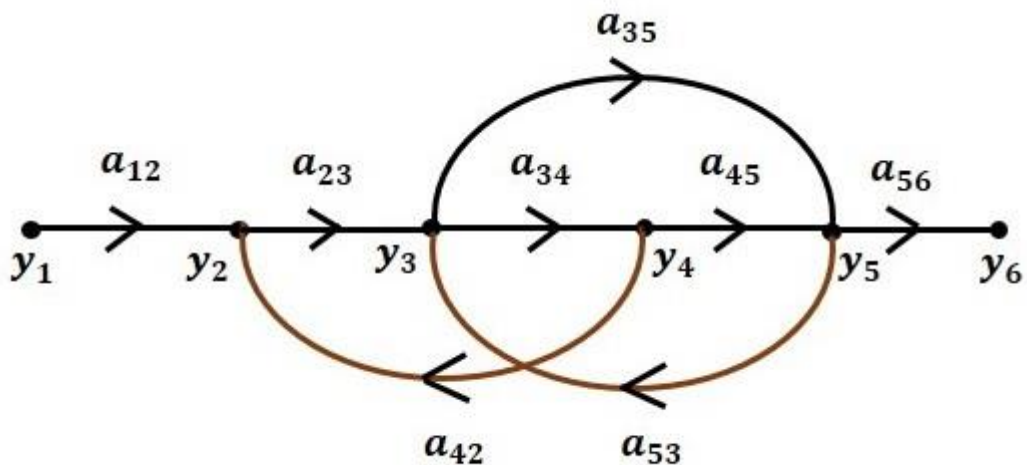
Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

- Represent all the signals, variables, summing points and take-off points of block diagram as nodes in signal flow graph.

- Represent the blocks of block diagram as branches in signal flow graph.

- Represent the transfer functions inside the blocks of block diagram as gains of the branches in signal flow graph.

- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. For example, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Example

Let us convert the following block diagram into its equivalent signal flow graph.
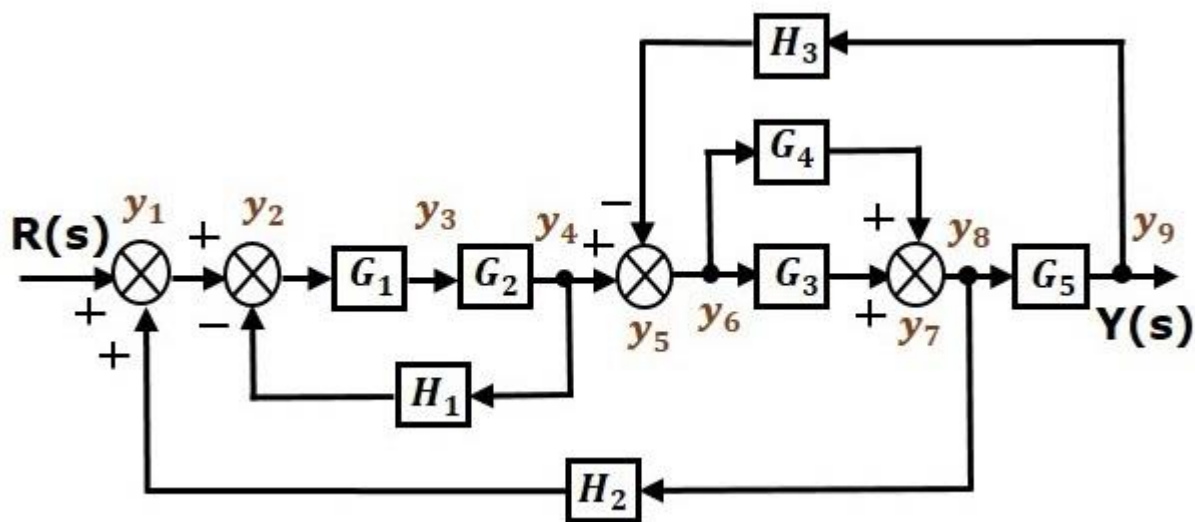
Represent the input signal R(s) and output signal C(s) of block diagram as input node R(s) and output node C(s) of signal flow graph.

Just for reference, the remaining nodes (y1 to y9) are labelled in the block diagram. There are nine nodes other than input and output nodes. That is four nodes for four summing points, four nodes for four take-off points and one node for the variable between blocks G1 and G2.

The following figure shows the equivalent signal flow graph.
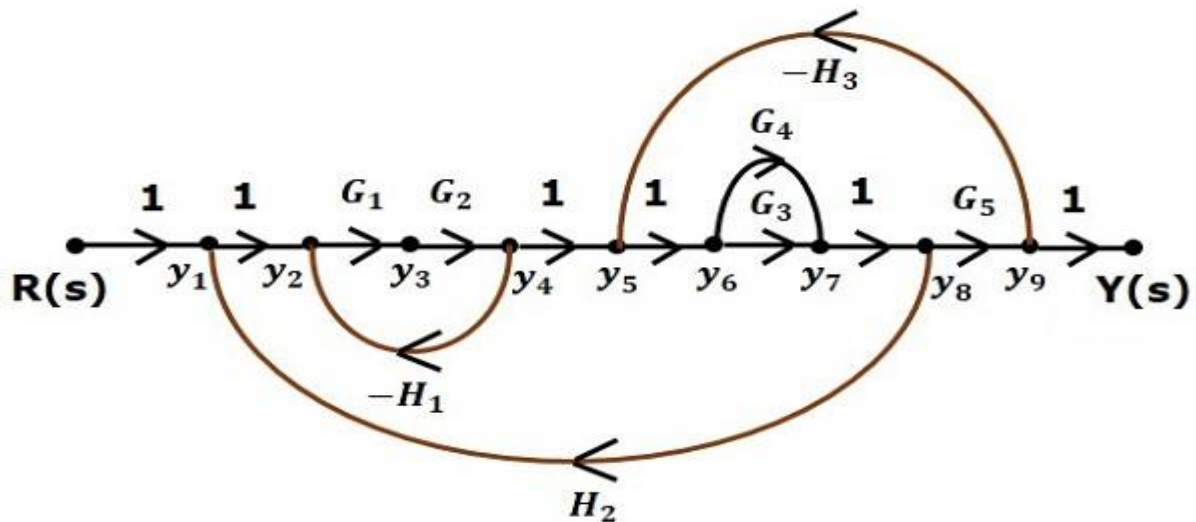
Reduction using Mason's gain formula

With the help of Mason's gain formula, we can calculate the transfer function of the signal flow graph. This is the advantage of signal flow graphs. Here, we no need to simplify (reduce) the signal flow graphs for calculating the transfer function.

Suppose there are 'N' forward paths in a signal flow graph. The gain between the input and the output nodes of a signal flow graph is nothing but the transfer function of the system. It can be calculated by using Mason's gain formula.

Mason's gain formula is,

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

Where,

- **C(s)** is the output node

- **R(s)** is the input node

- **T** is the transfer function or gain between R(s)R(s) and C(s)C(s)

- **P$_i$** is the ith forward path gain

$\Delta = 1$ − (sum of all individual loop gains) + (sum of gain products of all possible two non touching loops − (sum of gain products of all possible three non touching loops) +...

Δi is obtained from Δ by removing the loops which are touching the ith forward path.

Consider the following signal flow graph in order to understand the basic terminology involved here.
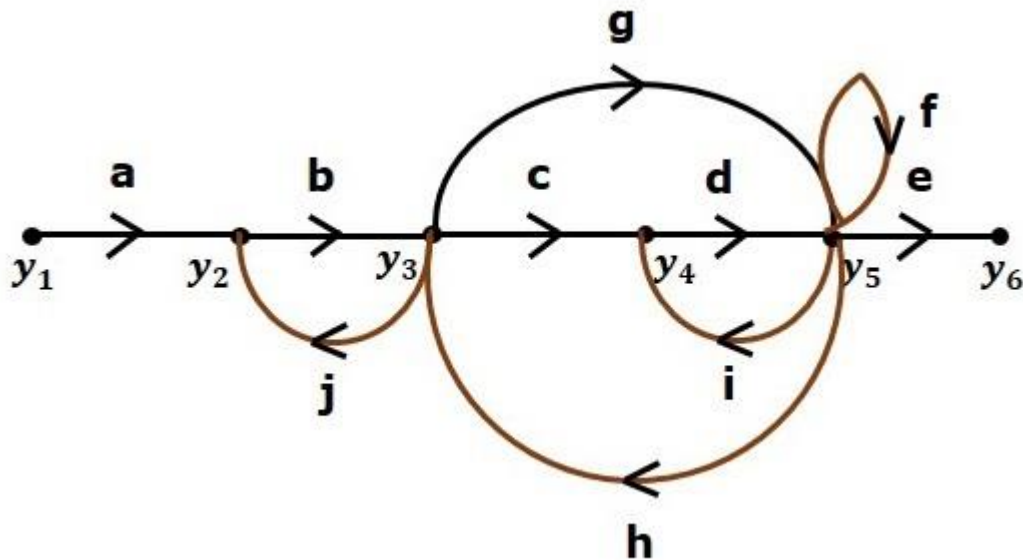
Path

It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples − y2→y3→y4→y5 and y5→y3→y2

Forward Path

The path that exists from the input node to the output node is known as forward path.

Examples − y1→y2→y3→y4→y5→y6 and y1→y2→y3→y5→y6.

Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples – abcde is the forward path gain of y1→y2→y3→y4→y5→y6 and abge is the forward path gain of y1→y2→y3→y5→y6.

Loop

The path that starts from one node and ends at the same node is known as loop. Hence, it is a closed path.

Examples − y2→y3→y2 and y3→y5→y3.

Loop Gain

It is obtained by calculating the product of all branch gains of a loop.

Examples − bj is the loop gain of y2→y3→y2 and gh is the loop gain of y3→y5→y3.

Non-touching Loops

These are the loops, which should not have any common node.

Examples −   The   loops, y2→y3→y2y2→y3→y2 and y4→y5→y4y4→y5→y4 are   non-touching.

Calculation of Transfer Function using Mason's Gain Formula

Let us consider the same signal flow graph for finding transfer function.
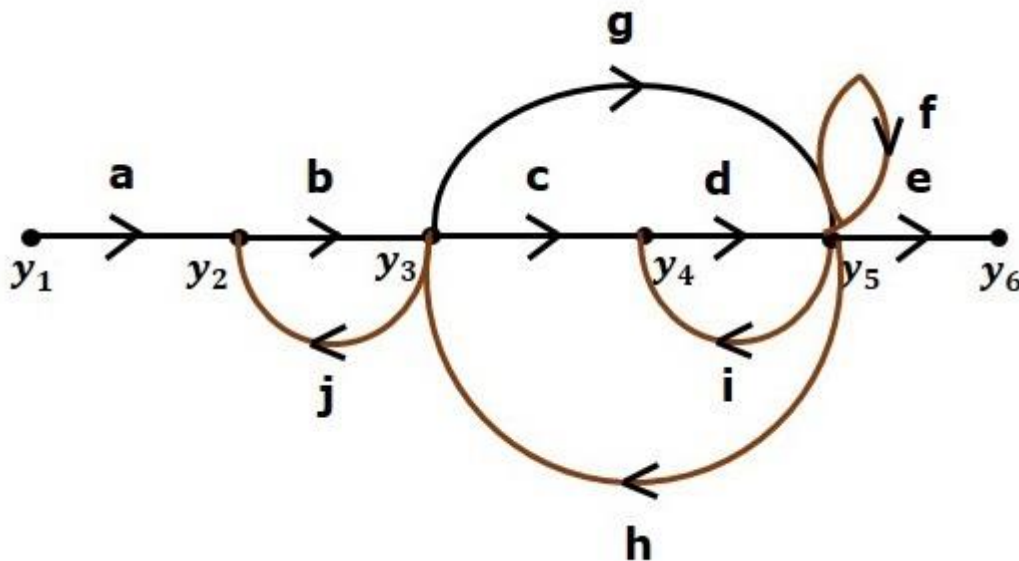
- Number of forward paths, N = 2.

- First forward path is - y1→y2→y3→y4→y5→y6.
- First forward path gain, p1=abcde.
- Second forward path is - y1→y2→y3→y5→y6.
- Second forward path gain, p2=abgep.

- Number of individual loops, L = 5.

- Loops are - y2→y3→y2, y3→y5→y3, y3→y4→y5→y3, y4→y5→y4 and y5→y5 .

- Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

- Number of two non-touching loops = 2.

- First non-touching loops pair is - y2→y3→y2, y4→y5→y4.

- Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$

- Second non-touching loops pair is - y2→y3→y2, y5→y5.

- Gain product of second non-touching loops pair is - $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta 1=1$.

Similarly, $\Delta 2=1$. Since, no loop which is non-touching to the second forward path.

Substitute, N = 2 in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^2 P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is –

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Synchros

The Synchro is a type of transducer which transforms the angular position of the shaft into an electric signal. It is used as an error detector and as a rotary position sensor. The error occurs in the system because of the misalignment of the shaft. The transmitter and the control transformer are the two main parts of the synchro.

Synchros Transmitter – Their construction is similar to the three phase alternator. The stator of the synchros is made of steel for reducing the iron losses. The stator is slotted for housing the three phase windings. The axis of the stator winding is kept 120º apart from each other.

Constructional Feature of Synchros Transmitter
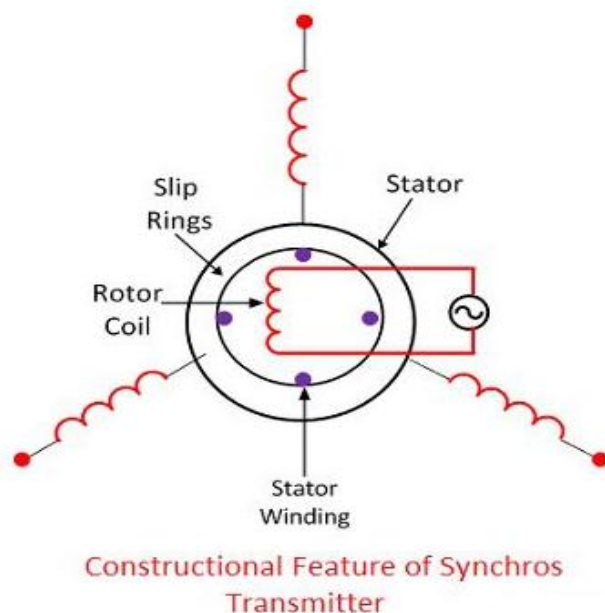
The AC voltage is applied to the rotor of the transmitter and it is expressed as

$$v_r = \sqrt{2} Sin\omega_c t$$

Where V_r – r.ms.value of rotor voltage

ω_c – carrier frequency

The coils of the stator windings are connected in star. The rotor of the synchros is a dumbbell in shape, and a concentric coil is wound on it. The AC voltage is applied to the rotor with the help of slip rings. The constructional feature of the synchros is shown in the figure below.

Sychro Transmitter
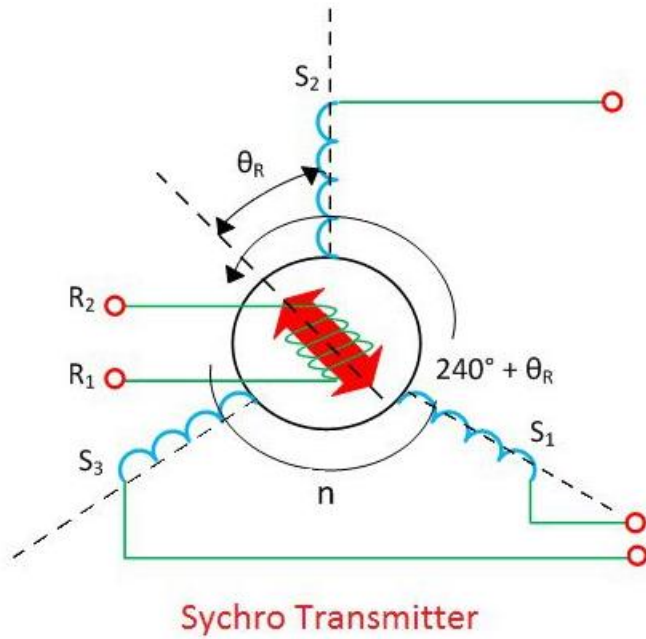
Consider the voltage is applied to the rotor of the transmitter as shown in the figure above.

Syncho Error Dectector
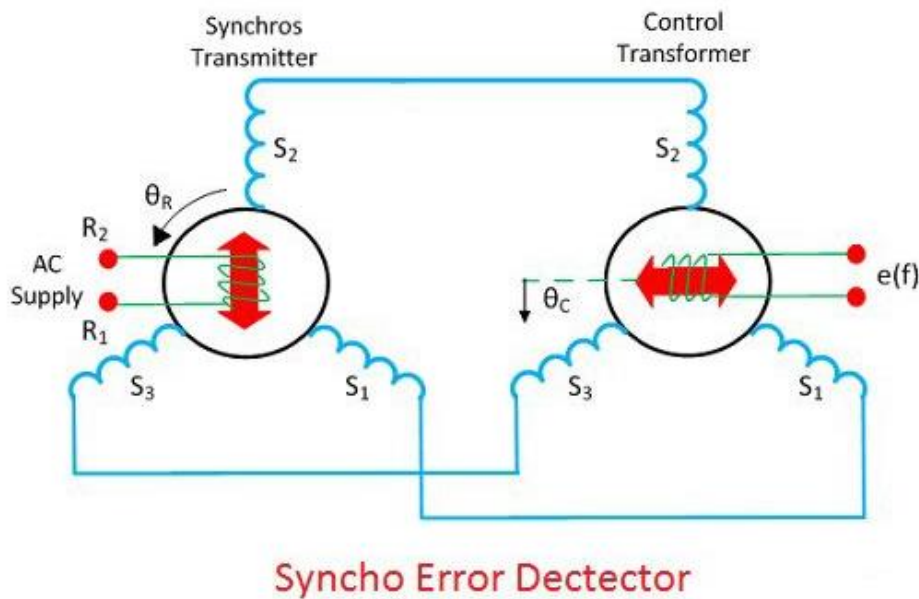
The voltage applied to the rotor induces the magnetizing current and an alternating flux along its axis. The voltage is induced in the stator winding because of the mutual induction between the rotor and stator flux. The flux linked in the stator winding is equal to the cosine of the angle between the rotor and stator. The voltage is induced in the stator winding.

Let V_{s1}, V_{s2}, V_{s3} be the voltages generated in the stator windings S_1, S_2, and S_3 respectively. The figure below shows the rotor position of the synchro transmitter. The rotor axis makes an angle θ_r concerning the stator windings S_2.

$$V_{s1n} = kV_r Sin\omega_c t cos(\theta_R + 120°)$$

$$V_{s2n} = kV_r Sin\omega_c t cos\theta_R$$

$$V_{s1n} = kV_r Sin\omega_c t cos(\theta_R + 240°)$$

The three terminals of the stator windings are

The variation in the stator terminal axis concerning the rotor is shown in the figure below.

$$e(t) = k'V_r Cos(90° - \theta_R + \theta_C)Sin\omega_c t$$

$$e(t) = k'V_r Sin(\theta_R - \theta_C)Sin\omega_c t$$

$$V_{s1s2} = V_{s1n} - V_{s2n}$$

$$V_{s1s2} = \sqrt{3}kV_r Sin(\theta_R + 240°)Sin\omega_c t$$

$$V_{s3s2} = V_{s2n} - V_{s3n}$$

$$V_{s1s2} = \sqrt{3}kV_r Sin(\theta_R + 120°)Sin\omega_c t$$

$$V_{s3s1} = V_{s3n} - V_{s1n}$$

$$V_{s3s1} = kV_r Sin\omega_c t Sin\theta_R$$

When the rotor angle becomes zero, the maximum current is produced in the stator windings S_2. The zero position of the rotor is used as a reference for determining the rotor angular position.

The output of the transmitter is given to stator winding of the control transformer which is shown in the above figure.

The current of the same and magnitude flow through the transmitter and control transformer of the synchros. Because of the circulating current, the flux is established between the air gap flux of the control transformer.

The flux axis of the control transformer and the transmitter is aligned in the same position. The voltage generates by the rotor of control transformer is equal to the cosine of the angle between the rotors of the transmitter and the controller. The voltage is given as

$$e(t) = k'V_r Cos\emptyset Sin\omega_c t$$

Where φ – angular displacement between the rotor axes of transmitter and controller.
Φ – 90° the axis between the rotor of transmitter and control transformer is perpendicular to each other. The above figure shows the zero position of the rotor of transmitter and receiver.

Consider the position of the rotor and the transmitter is changing in the same direction. An angle θ_R deflects the rotor of the transmitter and that of the control transformer is kept θ_C. The total angular separation between the rotors is $\Phi = (90° - \theta_R + \theta_C)$

The rotor terminal voltage of the Synchro transformer is given as

$$e(t) = k'V_r Cos(90° - \theta_R + \theta_C)Sin\omega_c t$$
$$e(t) = k'V_r Sin(\theta_R - \theta_C)Sin\omega_c t$$

The small angular displacement between their rotor position is given as

Sin $(\theta_R - \theta_C) = (\theta_R - \theta_C)$

On substituting the value of angular displacement in equation (1) we get

$$e(t) = k'V_r Cos\emptyset Sin\omega_c t$$

The synchro transmitter and the control transformer together used for detecting the error. The voltage equation shown above is equal to the shaft position of the rotors of control transformer and transmitter.
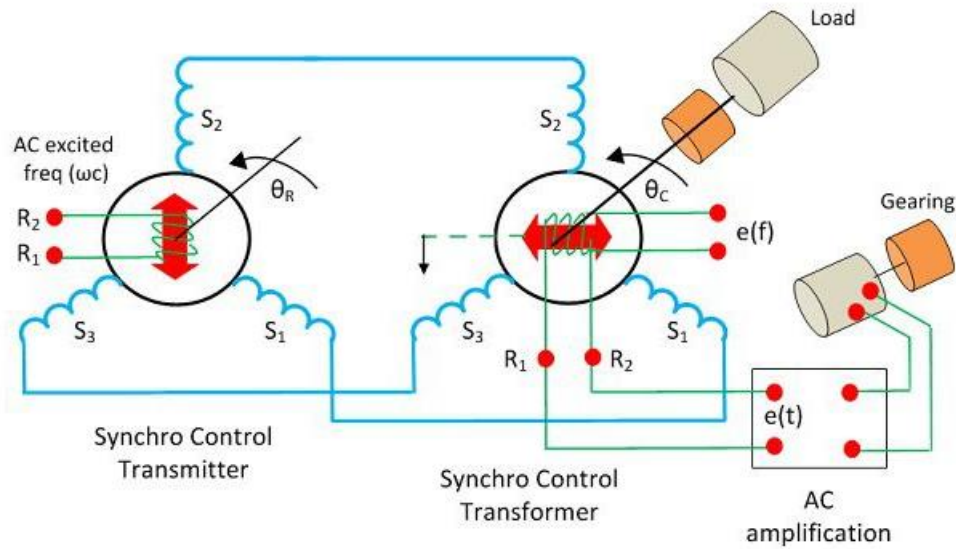
Positional Control System

The error signal is applied to the differential amplifier which gives input to the servo motor.

The gear of the servo motor rotates the rotor of the control transformer
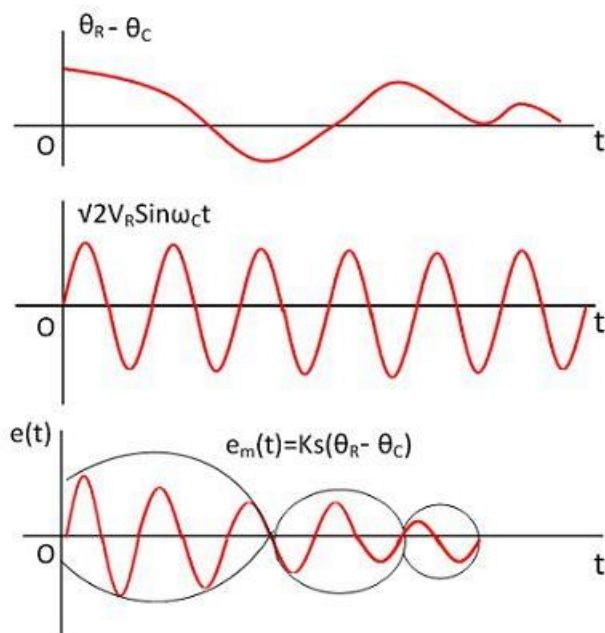
Waveform of Synchro Error Detector

The figure above shows the output of the synchro error detector which is a modulated signal.
The modulating wave above shown the misalignment between the rotor position and the
carrier wave.

$$e(t) = (\theta_R - \theta_C)$$

Where K_s is the error detector.

AC & DC servo motors

A servo motor is one of the widely used variable speed drives in industrial production and process automation and building technology worldwide. lthough servo motors are not a specific class of motor, they are intended and designed to use in motion control applications which require high accuracy positioning, quick reversing and exceptional performance.

These are widely used in robotics, radar systems, automated manufacturing systems, machine tools, computers, CNC machines, tracking systems, etc.,

A servo motor is a linear or rotary actuator that provides fast precision position control for closed-loop position control applications. Unlike large industrial motors, a servo motor is not used for continuous energy conversion.

Servo motors have a high speed response due to low inertia and are designed with small diameter and long rotor length. Then how do servo motors work?

Servo motors work on servo mechanism that uses position feedback to control the speed and final position of the motor. Internally, a servo motor combines a motor, feedback circuit, controller and other electronic circuit.

It uses encoder or speed sensor to provide speed feedback and position. This feedback signal is compared with input command position (desired position of the motor corresponding to a load), and produces the error signal (if there exist a difference between them).

The error signal available at the output of error detector is not enough to drive the motor. So the error detector followed by a servo amplifier raises the voltage and power level of the error signal and then turns the shaft of the motor to desired position.

Basically, servo motors are classified into AC and DC servo motors depending upon the nature of supply used for its operation. Brushed permanent magnet DC servo motors are used for simple applications owing to their cost, efficiency and simplicity.

These are best suited for smaller applications. With the advancement of microprocessor and power transistor, AC servo motors are used more often due to their high accuracy control.

DC Servo Motors

A DC servo motor consists of a small DC motor, feedback potentiometer, gearbox, motor drive electronic circuit and electronic feedback control loop. It is more or less similar to the normal DC motor.

The stator of the motor consists of a cylindrical frame and the magnet is attached to the inside of the frame. The rotor consists of brush and shaft. A commutator and a rotor metal supporting frame are attached to the outside of the shaft and the armature winding is coiled in the rotor metal supporting frame.

A brush is built with an armature coil that supplies the current to the commutator. At the back of the shaft, a detector is built into the rotor in order to detect the rotation speed.

With this construction, it is simple to design a controller using simple circuitry because the torque is proportional to the amount of current flow through the armature.

Working Principle of DC Servo Motor

A DC servo motor is an assembly of four major components, namely a DC motor, a position sensing device, a gear assembly, and a control circuit. The below figure shows the parts that consisting in RC servo motors in which small DC motor is employed for driving the loads at precise speed and position.

A DC reference voltage is set to the value corresponding to the desired output. This voltage can be applied by using another potentiometer, control pulse width to voltage converter, or through timers depending on the control circuitry.

The dial on the potentiometer produces a corresponding voltage which is then applied as one of the inputs to error amplifier.

In some circuits, a control pulse is used to produce DC reference voltage corresponding to desired position or speed of the motor and it is applied to a pulse width to voltage converter.

In this converter, the capacitor starts charging at a constant rate when the pulse high. Then the charge on the capacitor is fed to the buffer amplifier when the pulse is low and this charge is further applied to the error amplifier.

So the length of the pulse decides the voltage applied at the error amplifier as a desired voltage to produce the desired speed or position.

In digital control, microprocessor or microcontroller are used for generating the PWM pluses in terms of duty cycles to produce more accurate control signals.
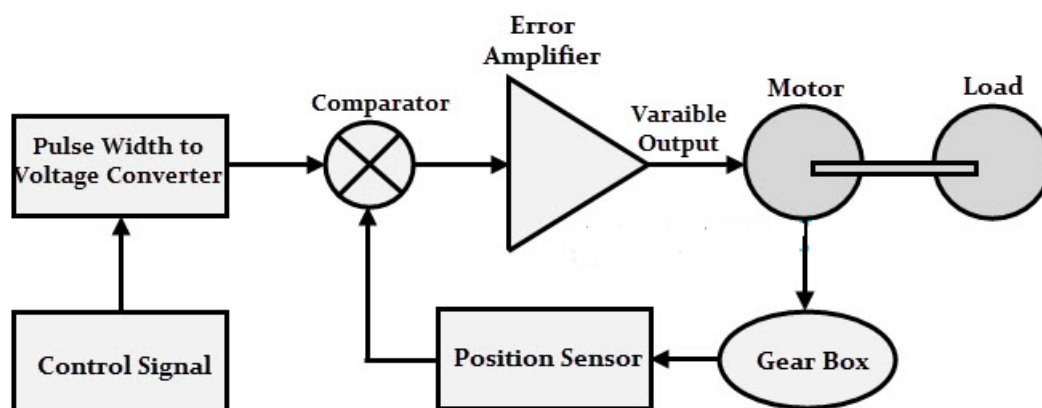
The feedback signal corresponding to the present position of the load is obtained by using a position sensor. This sensor is normally a potentiometer that produces the voltage corresponding to the absolute angle of the motor shaft through gear mechanism. Then the feedback voltage value is applied at the input of error amplifier (comparator).

The error amplifier is a negative feedback amplifier and it reduces the difference between its inputs. It compares the voltage related to current position of the motor (obtained by

potentiometer) with desired voltage related to desired position of the motor (obtained by pulse width to voltage converter), and produces the error either a positive or negative voltage.

This error voltage is applied to the armature of the motor. If the error is more, the more output is applied to the motor armature.

As long as error exists, the amplifier amplifies the error voltage and correspondingly powers the armature. The motor rotates till the error becomes zero. If the error is negative, the armature voltage reverses and hence the armature rotates in the opposite direction.

AC Servo Motors

AC servo motors are basically two-phase squirrel cage induction motors and are used for low power applications. Nowadays, three phase squirrel cage induction motors have been modified such that they can be used in high power servo systems.

The main difference between a standard split-phase induction motor and AC motor is that the squirrel cage rotor of a servo motor has made with thinner conducting bars, so that the motor resistance is higher.

Based on the construction there are two distinct types of AC servo motors, they are synchronous type AC servo motor and induction type AC servo motor.

Synchronous-type AC servo motor

It consist of stator and rotor. The stator consists of a cylindrical frame and stator core. The armature coil wound around the stator core and the coil end is connected to with a lead wire through which current is provided to the motor.

The rotor consists of a permanent magnet and hence they do not rely on AC induction type rotor that has current induced into it. And hence these are also called as brushless servo motors because of structural characteristics.

When the stator field is excited, the rotor follows the rotating magnetic field of the stator at the synchronous speed. If the stator field stops, the rotor also stops. With this permanent magnet rotor, no rotor current is needed and hence less heat is produced.

Also, these motors have high efficiency due to the absence of rotor current. In order to know the position of rotor with respect to stator, an encoder is placed on the rotor and it acts as a feedback to the motor controller.

The **induction-type AC servo moto**r structure is identical with that of general motor. In this motor, stator consists of stator core, armature winding and lead wire, while rotor consists of shaft and the rotor core that built with a conductor as similar to squirrel cage rotor.

The working principle of this servo motor is similar to the normal induction motor. Again the controller must know the exact position of the rotor using encoder for precise speed and position control.

Working Principle of AC Servo Motor

The schematic diagram of servo system for AC two-phase induction motor is shown in the figure below. In this, the reference input at which the motor shaft has to maintain at a certain position is given to the rotor of synchro generator as mechanical input theta. This rotor is connected to the electrical input at rated voltage at a fixed frequency.
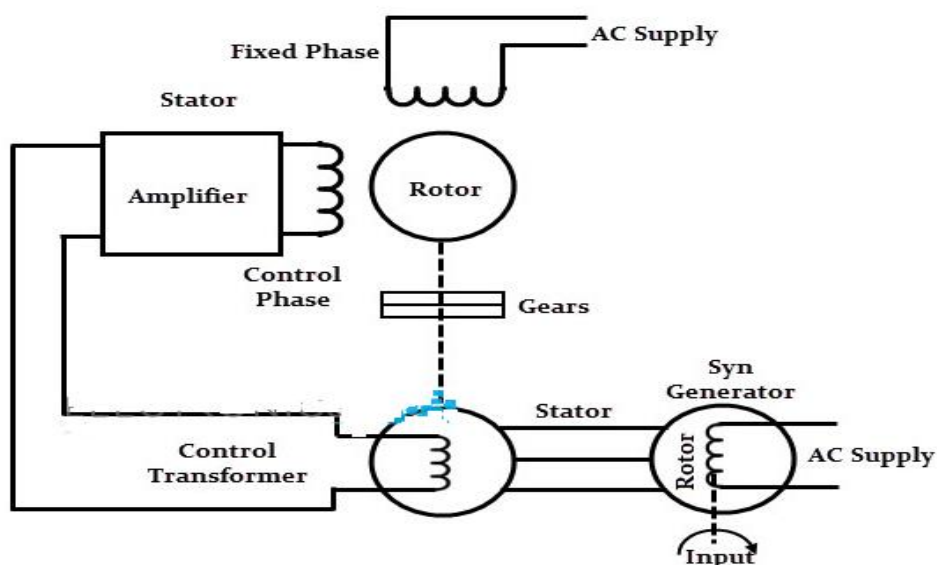
The three stator terminals of a synchro generator are connected correspondingly to the terminals of control transformer. The angular position of the two-phase motor is transmitted to the rotor of control transformer through gear train arrangement and it represents the control condition alpha.

Initially, there exist a difference between the synchro generator shaft position and control transformer shaft position. This error is reflected as the voltage across the control transformer. This error voltage is applied to the servo amplifier and then to the control phase of the motor.

With the control voltage, the rotor of the motor rotates in required direction till the error becomes zero. This is how the desired shaft position is ensured in AC servo motors.

Alternatively, modern AC servo drives are embedded controllers like PLCs, microprocessors and microcontrollers to achieve variable frequency and variable voltage in order to drive the motor.

Stepper motor

It is an electro mechanical digital device. It transforms electrical pulses in to equal increments of rotary shaft motion called steps. They function in conjunction with electronic switching devices.

It has a wound stator & a non excited rotor. The number of rotor and stator poles or teeth determines the size of the step called step angle.

Stepper motors are used in computer peripherals, X-Y plotters, scientific instruments, robots and machine tools.

Working Principle

Stepper motor is a brushless DC motor that rotates in steps. This is very useful because it can be precisely positioned without any feedback sensor, which represents an open-loop controller. The stepper motor consists of a rotor that is generally a permanent magnet and it is

surrounded by the windings of the stator. As we activate the windings step by step in a particular order and let a current flow through them they will magnetize the stator and make electromagnetic poles respectively that will cause propulsion to the motor. So that' the basic working principle of the stepper motors.

There are several different ways of driving the stepper motor. The first one is the Wave Drive or Single-Coil Excitation. In this mode we active just one coil at a time which means that for this example of motor with 4 coils, the rotor will make full cycle in 4 steps.
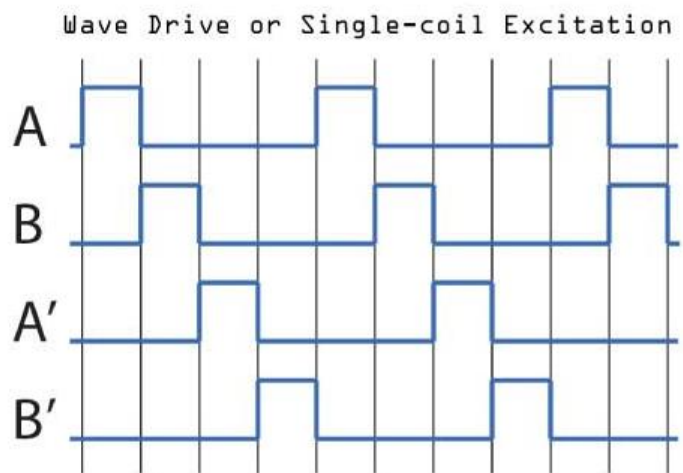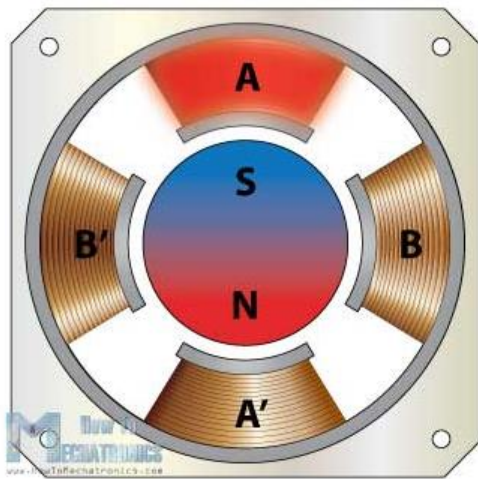
MODULE II

TIME RESPONSE ANALYSIS

Standard test signals

Introduction

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.
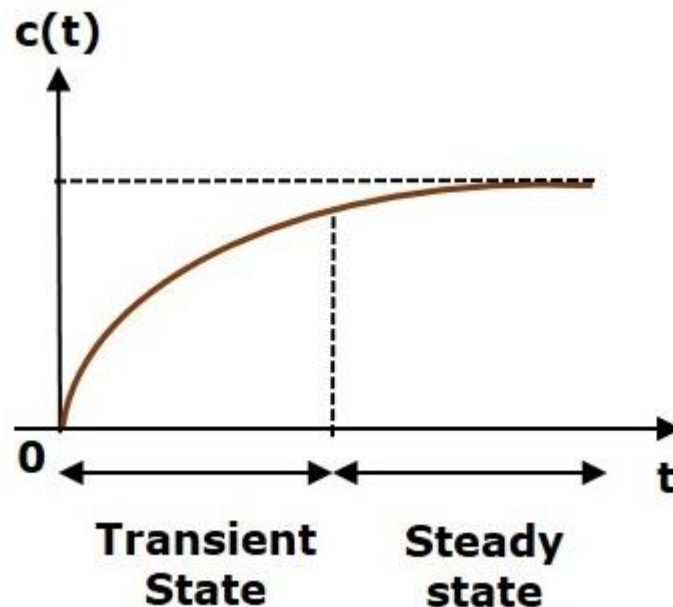
Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t \to \infty} c_{tr}(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system

$$c(t) = 10 + 5e^{-t}$$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

and $\int_{0^-}^{0^+} \delta(t)dt = 1$

The following figure shows unit impulse signal.
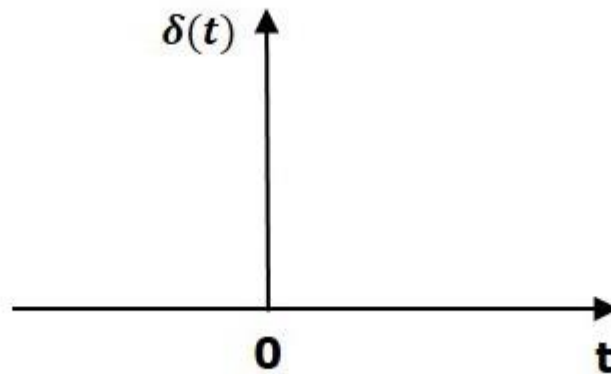
So, the unit impulse signal exists only at 't' is equal to zero. The area of this signal under small interval of time around 't' is equal to zero is one. The value of unit impulse signal is zero for all other values of 't'.

Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t) = 1; \ t \geq 0$$

$$= 0; \ t < 0$$

Following figure shows unit step signal.
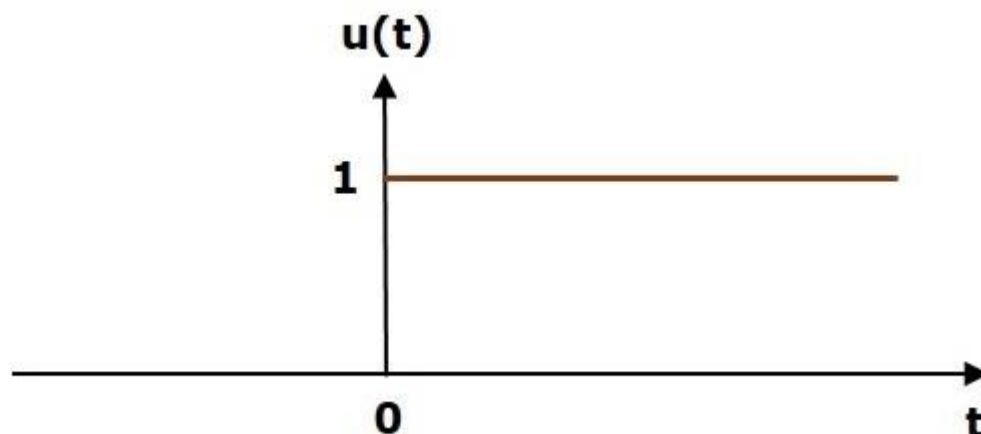
So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of 't'.

Unit Ramp Signal

A unit ramp signal, r(t) is defined as

$$r(t) = t; \quad t \geq 0$$

$$= 0; \quad t < 0$$

We can write unit ramp signal, r(t)r(t) in terms of unit step signal, u(t)u(t) as

$$r(t) = t\, u(t)$$

Following figure shows unit ramp signal.
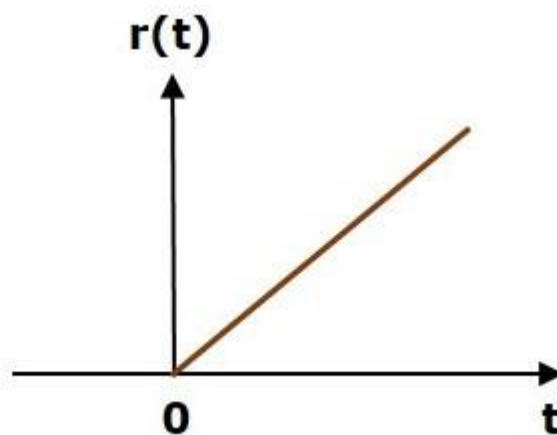
So, the unit ramp signal exists for all positive values of 't' including zero. And its value increases linearly with respect to 't' during this interval. The value of unit ramp signal is zero for all negative values of 't'.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

$$p(t) = \frac{t^2}{2}; t \geq 0$$

$$= 0; t < 0$$

We can write unit parabolic signal, p(t) in terms of the unit step signal, u(t) as,

$$p(t) = \frac{t^2}{2} u(t)$$

The following figure shows the unit parabolic signal.
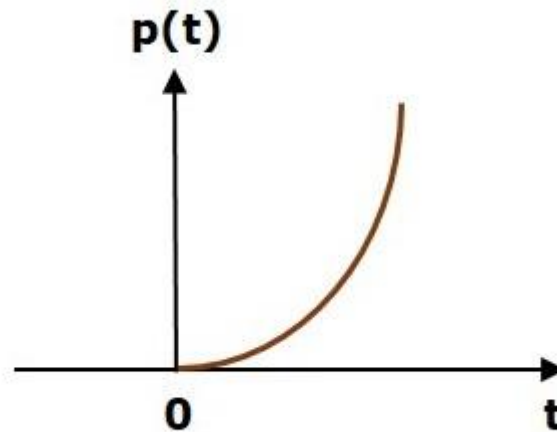
So, the unit parabolic signal exists for all the positive values of **'t'** including zero. And its value increases non-linearly with respect to 't' during this interval. The value of the unit parabolic signal is zero for all the negative values of 't'.

Time response of first order systems

Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.
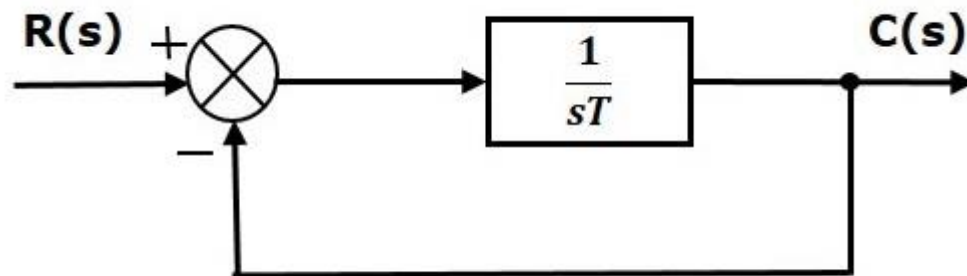
We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, G(s)=1/sT in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT + 1}\right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),

- **R(s)** is the Laplace transform of the input signal r(t), and

- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal r(t).
- Consider the equation, C(s)=(1sT+1)R(s)
- Substitute R(s) value in the above equation.
- Do partial fractions of C(s) if required.
- Apply inverse Laplace transform to C(s).

Impulse Response of First Order System

Consider the unit impulse signal as an input to the first order system.

So, r(t)=δ(t) Apply Laplace transform on both the sides.
R(s)=1, Consider the equation,

$$C(s) = \left(\frac{1}{sT + 1}\right) R(s)$$

Substitute, R(s)=1R(s)=1 in the above equation.

$$C(s) = \left(\frac{1}{sT + 1}\right)(1) = \frac{1}{sT + 1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

Apply inverse Laplace transform on both sides.

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

The unit impulse response is shown in the following figure.
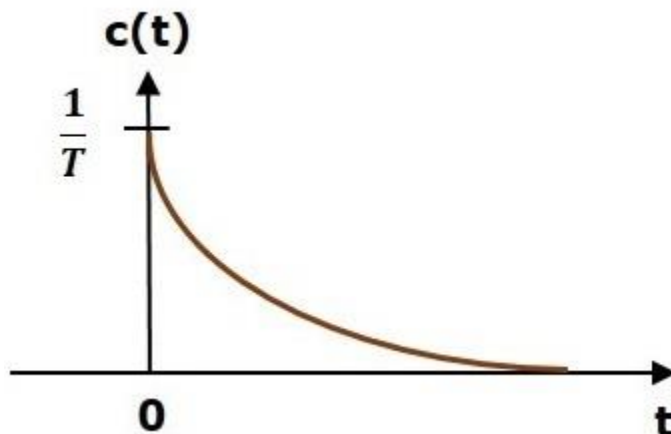
The **unit impulse response**, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, r(t)=u(t)

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

$$R(s) = \frac{1}{s}$$

Substitute,   in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s(sT+1)} = \frac{A(sT+1)+Bs}{s(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0=T+B \Rightarrow B=-T$$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

$$c_{ss}(t) = u(t)$$

The steady state term in the unit step response is -

The following figure shows the unit step response.
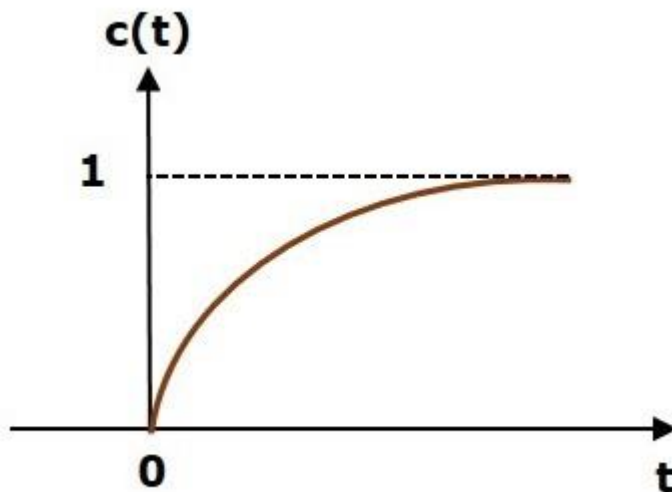
The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

So, r(t)=t u(t)

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Similarly, substitute B = −T and equate the coefficient of s^2 terms on both the sides. You will get C= T^2

Substitute A = 1, B = −T and C= T^2 in the partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)} \right) u(t)$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

The transient term in the unit ramp response is –

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit ramp response is –

$$c_{ss}(t) = (t - T)u(t)$$

The following figure shows the unit ramp response.
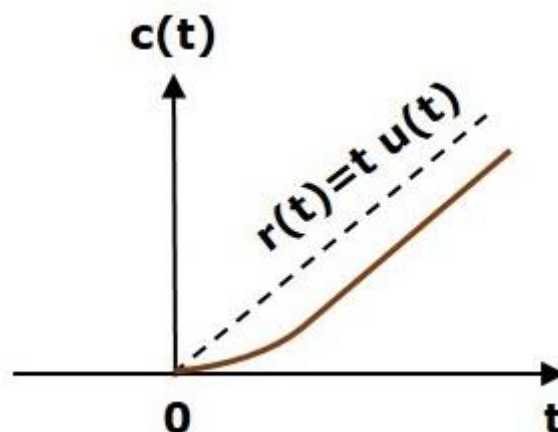
The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the unit parabolic signal as an input to the first order system.

$$r(t) = \frac{t^2}{2}u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of C(s)

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying,

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The unit parabolic response, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is

$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit parabolic response is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

Characteristic Equation of Feedback control systems

The control ratio is the closed loop transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

The denominator of closed loop transfer function determines the characteristic equation of the system.

Which is usually determined as:

$$1 \pm G(s)H(s) = 0$$

Transient response of second order systems

Consider the following block diagram of closed loop control system. Here, an open loop transfer function is connected with a unity negative feedback.
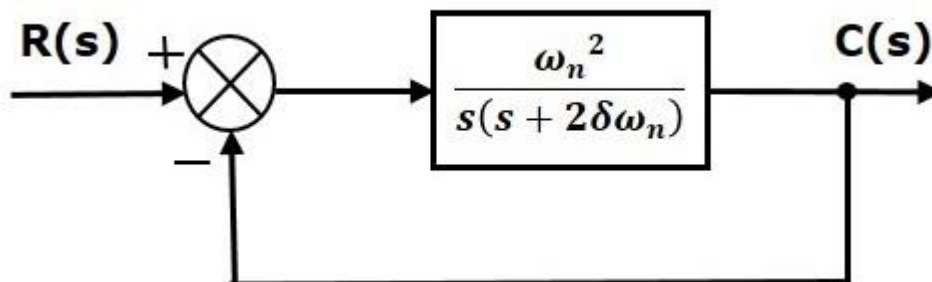
We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are –

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The two roots are imaginary when $\delta = 0$.

- The two roots are real and equal when $\delta = 1$.

- The two roots are real but not equal when $\delta > 1$.

- The two roots are complex conjugate when $0 < \delta < 1$.

We can write C(s) equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: $\delta = 0$

Substitute, $\delta=0\delta=0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when /delta=0/delta=0 will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, /delta=1/delta=1 in the transfer function.

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of C(s)

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)\right)}$$

Do partial fractions of C(s),

$$C(s) = \frac{\omega_n^2}{s\left((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

After simplifying, you will get the values of A, B and C as 1,−1and−2δωn1respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} - \frac{\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ωdωd in the above equation

$$C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1 - \delta^2}}\left(\frac{\omega_d}{(s + \delta\omega_n)^2 + \omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1 - \delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}}\left((\sqrt{1 - \delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right)u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right)u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}}\right)\sin(\omega_d t + \theta)\right)u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right) R(s)$$

Substitute, R(s)=1/s in the above equation,

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of C(s)

$$C(s) = \frac{\omega_n^2}{s(s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1})(s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1})}$$

$$= \frac{A}{s} + \frac{B}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}} + \frac{C}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}$$

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\left(\frac{1}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}}\right)$$

$$- \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\right)\left(\frac{1}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right) e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right) e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right) u$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

Time domain specifications

The step response of the second order system for the underdamped case is shown in the following figure.
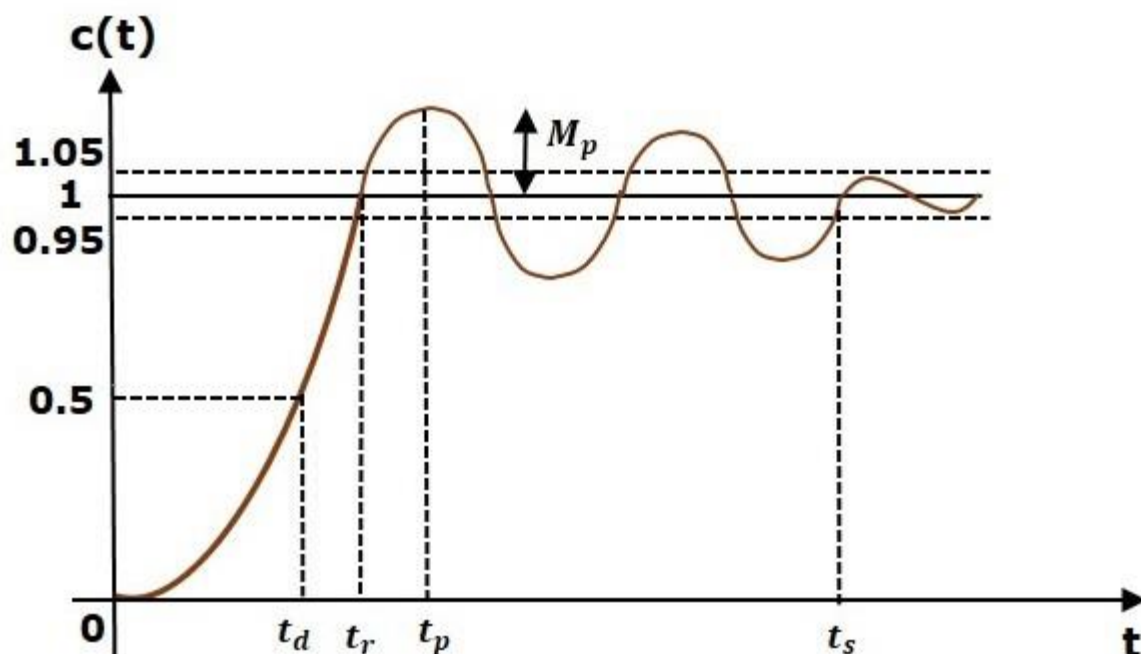
All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by tdtd.

Consider the step response of the second order system for $t \geq 0$, when 'δ' lies between zero and one.

$$t_d = \frac{1+0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by t_r.

$$t_r = \frac{\pi - \theta}{\omega_d}$$

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by tp

$$t_p = \frac{\pi}{\omega_d}$$

Peak Overshoot

Peak overshoot M_p is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

$$M_p = c(t_p) - c(\infty)$$

$$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} \right) \times 100\%$$

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by tsts.

The settling time for 5% tolerance band is –

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is –

$$t_s = \frac{4}{\delta \omega_n} = 4\tau$$

Steady state response

A steady-state response is the behavior of a circuit after a long time when steady conditions have been reached after an external excitation. A steady-state response is the behavior of a circuit after a long time when steady conditions have been reached after an external excitation. ... The poles and zeros will control the steady-state response at any given frequency.

Steady state errors and error constants

The deviation of the output of control system from desired response during steady state is known as steady state error. It is represented as essss. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Where,

E(s) is the Laplace transform of the error signal, e(t)

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.
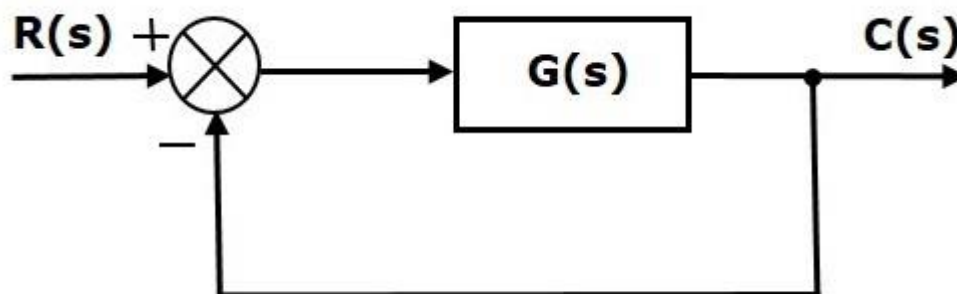
Where,

- R(s) is the Laplace transform of the reference Input signal r(t)r(t)
- C(s) is the Laplace transform of the output signal c(t)c(t)

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1 + G(s)}$$

The output of the summing point is –

$$E(s) = R(s) - C(s)$$

Substitute C(s) value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1 + G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

Substitute E(s) value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \to 0} s^2 G(s)$

Where, Kp, Kv and Ka are position error constant, velocity error constant and acceleration error constant respectively.

Effects of proportional derivative systems

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides.

$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is KDs.

Where, KD is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.
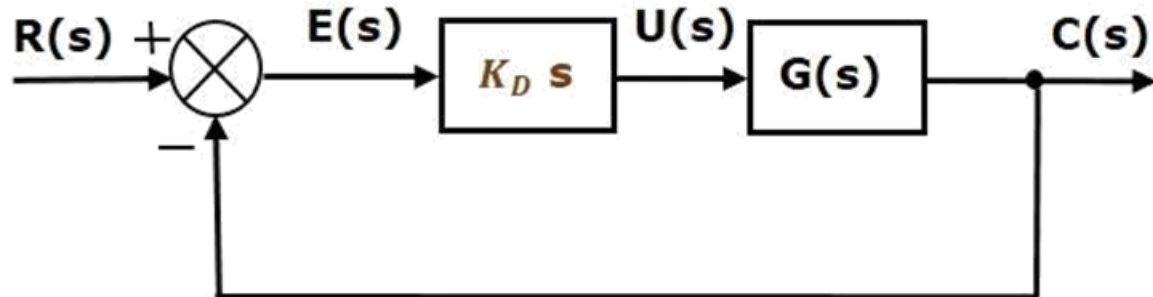
The derivative controller is used to make the unstable control system into a stable one.

Effects of proportional integral systems

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t)dt$$

Apply Laplace transform on both the sides -

$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where, KI is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.
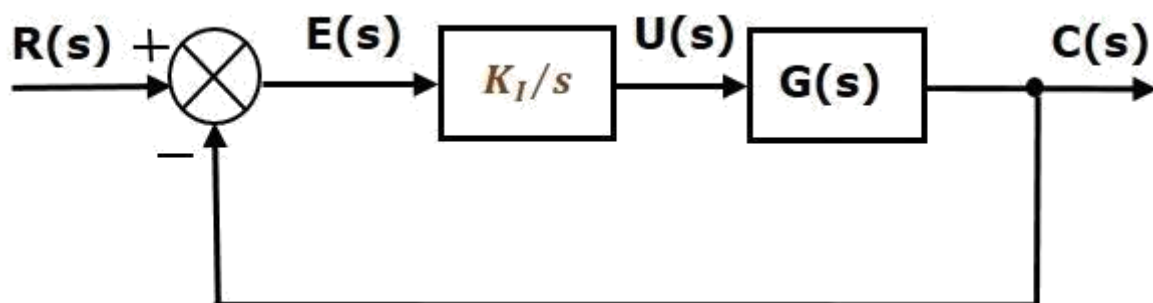
The integral controller is used to decrease the steady state error.

MODULE III

STABILITY ANALYSIS IN S-DOMAIN

A:

The concept of stability

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.
A **stable system** produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.
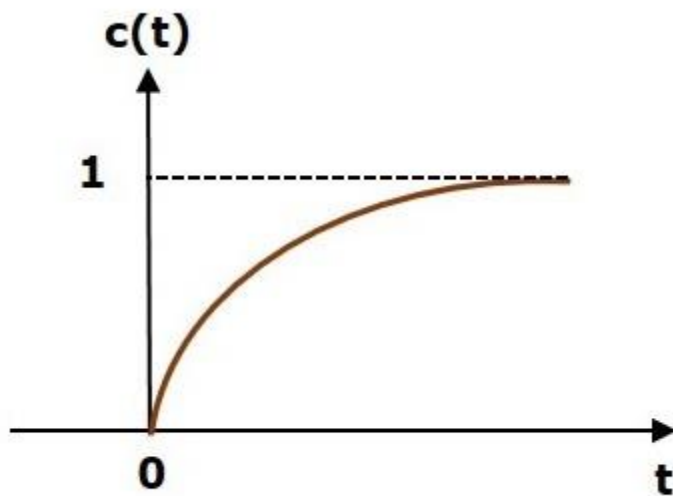
This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of **t** including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability

We can classify the systems based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the absolutely stable system. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of 's' plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

Routh's stability criterion– Qualitative stability and conditional stability

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Consider the characteristic equation of the order 'n' is -

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Note that, there should not be any term missing in the **n^th** order characteristic equation. This means that the **n^th** order characteristic equation should not have any coefficient that is of zero value.

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method

If all the roots of the characteristic equation exist to the left half of the 's' plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the 's' plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable.

Follow this procedure for forming the Routh table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of sn and continue up to the coefficient of s0.
- Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of **row** s0 is an. Here, an is the coefficient of s0 in the characteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6	...	...
s^{n-1}	a_1	a_3	a_5	a_7	...	...
s^{n-2}	b_1 $= \frac{a_1 a_2 - a_3 a_0}{a_1}$	b_2 $= \frac{a_1 a_4 - a_5 a_0}{a_1}$	b_3 $= \frac{a_1 a_6 - a_7 a_0}{a_1}$	...	...	...
s^{n-3}	c_1 $= \frac{b_1 a_3 - b_2 a_1}{b_1}$	c_2 $= \frac{b_1 a_5 5 - b_3 a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4 + 3s^3 + 3s^2 + 2s + 1$ are positive. So, the control system satisfies the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\dfrac{(3\times3)-(2\times1)}{3}=\dfrac{7}{3}$	$\dfrac{(3\times1)-(0\times1)}{3}=\dfrac{3}{3}=1$	
s^1	$\dfrac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}}=\dfrac{5}{7}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

Special Cases of Routh Array

We may come across two types of situations, while forming the Routh table. It is difficult to complete the Routh table from these two situations.

The two special cases are −

- The first element of any row of the Routh array is zero.
- All the elements of any row of the Routh array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

First Element of any row of the Routh array is zero

If any row of the Routh array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, $\epsilon\epsilon$. And then continue the process of completing the Routh table. Now, find the number of sign changes in the first column of the Routh table by substituting $\epsilon\epsilon$ tends to zero.

Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4 + 2s^3 + s^2 + 2s + 1$

are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	1	1
s^3	2̶ 1	2̶ 1	
s^2	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(0\times1)}{1} = 1$	
s^1			
s^0			

The row s^3 elements have 2 as the common factor. So, all these elements are divided by 2.

Special case (i) − Only the first element of row s^2 is zero. So, replace it by $\epsilon\epsilon$ and continue the process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon \times 1)-(1 \times 1)}{\epsilon} = \frac{\epsilon-1}{\epsilon}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

As $\epsilon\epsilon$ tends to zero, the Routh table becomes like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

All the Elements of any row of the Routh array are zero

In this case, follow these two steps −

- Write the auxilary equation, A(s) of the row, which is just above the row of zeros.

- Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example

Let us find the stability of the control system having characteristic equation,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^5	1	1	1
s^4	3 1	3 1	3 1
s^3	$\frac{(1\times1)-(1\times1)}{1}=0$	$\frac{(1\times1)-(1\times1)}{1}=0$	
s^2			
s^1			
s^0			

The row s^4 elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) − All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4.

$$A(s) = s^4 + s^2 + 1$$

Differentiate the above equation with respect to s.

$$\frac{dA(s)}{ds} = 4s^3 + 2s$$

Place these coefficients in row s^3.

s^5	1	1	1
s^4	1	1	1
s^3	4 2	2 1	
s^2	$\frac{(2\times1)-(1\times1)}{2} = 0.5$	$\frac{(2\times1)-(0\times1)}{2} = 1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5} = \frac{-1.5}{0.5} = -3$		
s^0	1		

step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

Limitations of Routh's stability

The limitations of the Routh-Hurwitz stability criteria are

(1) It is valid only if the Characteristic equation is algebraic.

(2) If any co-efficient of the Characteristic equation is complex or contains power of 'e' this criterion can not be applied.

(3) It gives information about how many roots are lying in the RHS of S-plane; values of the roots are not available. Also it cannot distinguish between real & complex roots.

B: Root Locus Technique

The root locus concept

The root locus is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs.

The Root locus is the locus of the roots of the characteristic equation by varying system gain K from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1+G(s)H(s)=0$$

We can represent G(s)H(s) as,

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

Where,

- K represents the multiplying factor

- N(s) represents the numerator term having (factored) n^{th} order polynomial of 's'.

- D(s) represents the denominator term having (factored) m^{th} order polynomial of 's'.

Substitute, G(s)H(s)G(s)H(s) value in the characteristic equation.

$$1 + k\frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

Case 1 − K = 0

If K=0, then D(s)=0.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 − K = ∞

Re-write the above characteristic equation as

$$K\left(\frac{1}{K} + \frac{N(s)}{D(s)}\right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute, K=∞K=∞ in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

f K=∞, then N(s)=0. It means the closed loop poles are equal to the open loop zeros when K is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

Angle Condition and Magnitude Condition

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of G(s)H(s) is

$$\angle G(s)H(s) = \tan^{-1}\left(\frac{0}{-1}\right) = (2n+1)\pi$$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180^0.

Magnitude of G(s)H(s) is −

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

Construction of root loci

Follow these rules for constructing a root locus.

Rule 1 − Locate the open loop poles and zeros in the 's' plane.

Rule 2 − Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N = P \quad \text{if} \quad P \geq Z$$

$$N = Z \quad \text{if} \quad P < Z$$

Rule 3 − Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule 4 − Find the centroid and the angle of asymptotes.

- If P=Z, then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If P>Z, then Z number of root locus branches start at finite open loop poles and end at finite open loop zeros and P−Z number of root locus branches start at finite open loop poles and end at infinite open loop zeros.
- If P<Z , then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and Z−P number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when P≠Z. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid α** by using this formula,

$$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P-Z}$$

The formula for the angle of **asymptotes θ** is

$$\theta = \frac{(2q+1)180^0}{P-Z}$$

Where,

$$q = 0, 1, 2, \ldots, (P-Z)-1$$

Rule 5 − Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.

- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.

- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rule 6 − Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.

- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note − Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write KK in terms of ss from the characteristic equation 1+G(s)H(s)=0.
- Differentiate K with respect to s and make it equal to zero. Substitute these values of ss in the above equation.
- The values of ss for which the K value is positive are the **break points**.

Rule 7 − Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕd is

$$\phi d = 180^0 - \phi$$

The formula for the **angle of arrival** ϕa is

$$\phi_a = 180^0 + \phi$$

Where,

$$\phi = \sum \phi_P - \sum \phi_Z$$

Example

Let us now draw the root locus of the control system having open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

Step 1 − The given open loop transfer function has three poles at s=0,s=−1 and s=−5. It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.

N=P=3
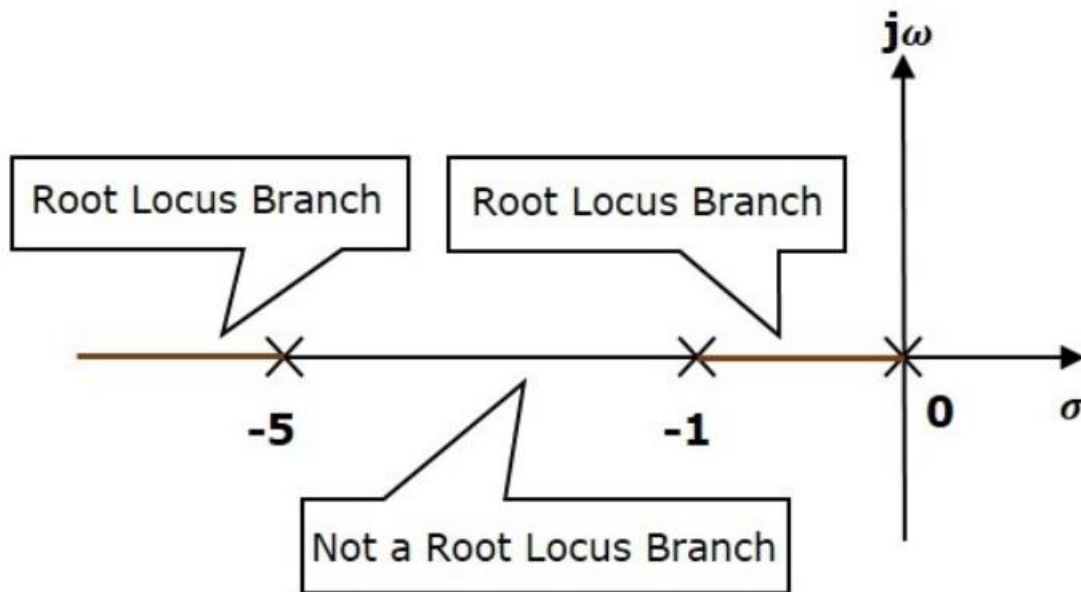
The three poles are located are shown in the above figure. The line segment between s=−1 and s=0 is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of s=−5.

Step 2 − We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid α=−2

The angle of asymptotes are θ=60^0,180^0 and 300^0.

The centroid and three asymptotes are shown in the following figure.
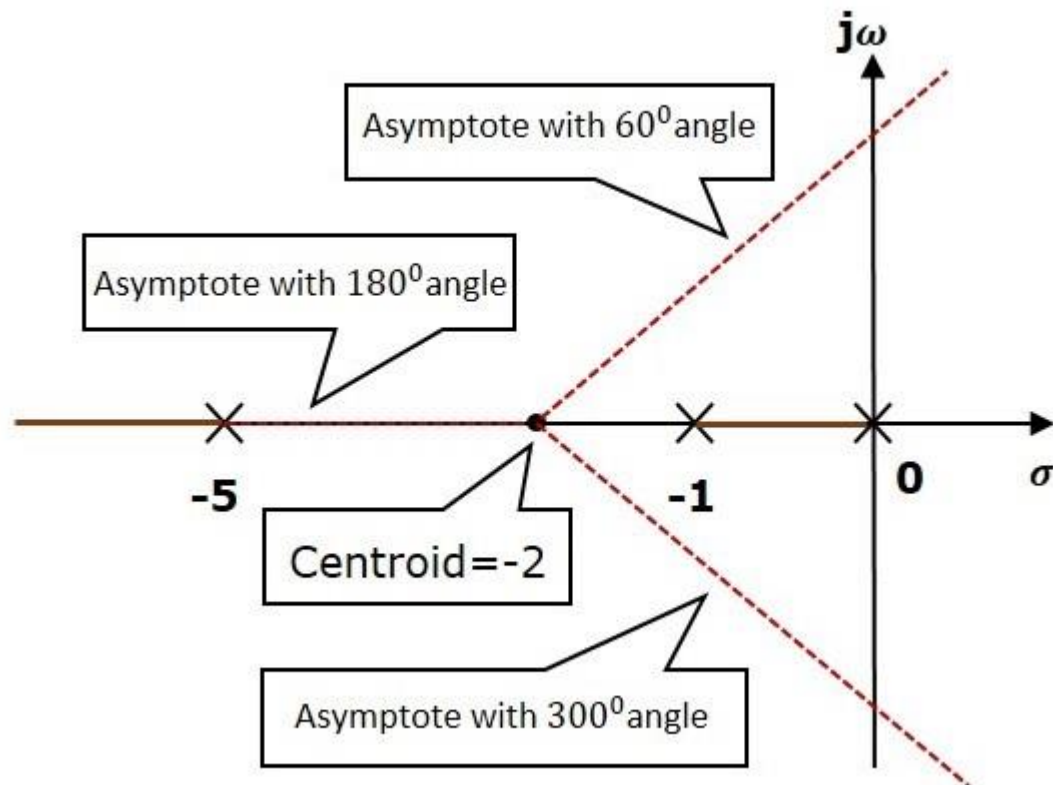
Step 3 − Since two asymptotes have the angles of 60^0 and 300^0, two root locus branches intersect the imaginary axis. By using the Routh array method and special case (ii), the root locus branches intersects the imaginary axis at $\sqrt{j5}$ and $\sqrt{-j5}$.

There will be one break-away point on the real axis root locus branch between the poles s=−1and s=0. By following the procedure given for the calculation of break-away point, we will get it as s=−0.473.

The root locus diagram for the given control system is shown in the following figure.
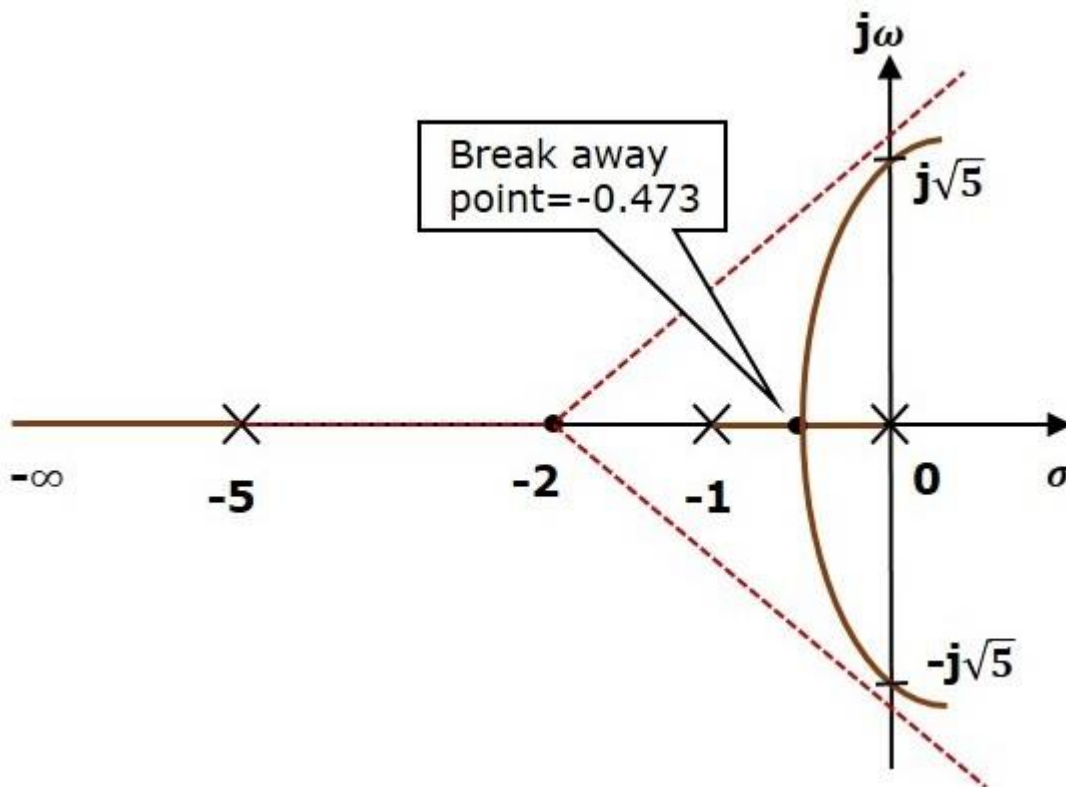
In this way, you can draw the root locus diagram of any control system and observe the movement of poles of the closed loop transfer function.

From the root locus diagrams, we can know the range of K values for different types of damping.

Effects of adding poles and zeros to G(s) H(s) on the root loci

The root locus can be shifted in **'s' plane** by adding the open loop poles and the open loop zeros.

- If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of 's' plane. Because of this, the damping ratio $\delta\delta$ decreases. Which implies, damped frequency $\omega d\omega d$ increases and the time domain specifications like delay time $tdtd$, rise time $trtr$ and peak time $tptp$ decrease. But, it effects the system stability.

- If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of 's' plane. So, it will increase the control system stability. In this case, the damping ratio $\delta\delta$ increases. Which implies, damped

frequency ω_d decreases and the time domain specifications like delay time t_d, rise time t_r and peak time t_p increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

MODULE IV

FREQUENCY RESPONSE ANALYSIS

Introduction

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steady state response of a system for an input sinusoidal signal is known as the **frequency response**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

Let the input signal be −

$$r(t) = A\sin(\omega_0 t)$$

The open loop transfer function will be −

$$G(s) = G(j\omega)$$

We can represent $G(j\omega)G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

Substitute, $\omega = \omega_0 \omega = \omega_0$ in the above equation.

$$G(j\omega_0) = |G(j\omega_0)| \angle G(j\omega_0)$$

The output signal is

$$c(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega = \omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega = \omega_0$.

Where,

- **A** is the amplitude of the input sinusoidal signal.

- **ω_0** is angular frequency of the input sinusoidal signal.

We can write, angular frequency $\omega_0 \omega_0$ as shown below.

$$\omega_0 = 2\pi f_0$$

Here, $f_0 f_0$ is the frequency of the input sinusoidal signal. Similarly, you can follow the same procedure for closed loop control system.

Frequency domain specifications

The frequency domain specifications are **resonant peak, resonant frequency and bandwidth**.

Consider the transfer function of the second order closed loop control system as,

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Substitute, $s=j\omega$ $s=j\omega$ in the above equation.

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow T(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega\omega_n + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\omega/\omega_n = u$ Substitute this value in the above equation,

$$T(j\omega) = \frac{1}{(1 - u^2) + j(2\delta u)}$$

Magnitude of T(jω) is –

$$\angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1 - u^2}\right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ωr. At ω=ωr, the first derivate of the magnitude of T(jω) is zero.

Differentiate M with respect to u.

$$\frac{dM}{du} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[2(1-u^2)(-2u) + 2(2\delta u)(2\delta)\right]$$

$$\Rightarrow \frac{dM}{du} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[4u(u^2 - 1 + 2\delta^2)\right]$$

Substitute, u=ur and $\frac{dM}{du} == 0$ in the above equation,

$$0 = -\frac{1}{2}\left[(1-u_r^2)^2 + (2\delta u_r)^2\right]^{-\frac{3}{2}}\left[4u_r(u_r^2 - 1 + 2\delta^2)\right]$$

$$\Rightarrow 4u_r(u_r^2 - 1 + 2\delta^2) = 0$$

$$\Rightarrow u_r^2 - 1 + 2\delta^2 = 0$$

$$\Rightarrow u_r^2 = 1 - 2\delta^2$$

$$\Rightarrow u_r = \sqrt{1 - 2\delta^2}$$

Substitute, $u_r = \frac{\omega_r}{\omega_n}$ in the above equation

$$\frac{\omega_r}{\omega_n} = \sqrt{1 - 2\delta^2}$$

$$\Rightarrow \omega_r = \omega_n\sqrt{1 - 2\delta^2}$$

Resonant Peak

It is the peak (maximum) value of the magnitude of T(jω). It is denoted by Mr.

At u=ur, the Magnitude of T(jω) is -

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1-2\delta^2}$ and $1 - u_r^2 = 2\delta^2$ in the above equation

$$M_r = \frac{1}{\sqrt{(2\delta^2)^2 + (2\delta\sqrt{1-2\delta^2})^2}}$$

$$\Rightarrow M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio δ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth

It is the range of frequencies over which, the magnitude of T(jω) drops to 70.7% from its zero frequency value.

At ω=0, the value of u will be zero.

Substitute, u=0 in M.

$$M = \frac{1}{\sqrt{(1-0^2)^2 + (2\delta(0))^2}} = 1$$

Therefore, the magnitude of T(jω) is one at ω=0.

At 3-dB frequency, the magnitude of T(jω) will be 70.7% of magnitude of T(jω) at ω=0.

$$\omega = \omega_B, M = 0.707(1) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\delta u_b)^2}}$$

$$\Rightarrow 2 = (1-u_b^2)^2 + (2\delta)^2 u_b^2$$

$$u_b^2 = x$$

$$\Rightarrow 2 = (1 - x)^2 + (2\delta)^2 x$$

$$\Rightarrow x^2 + (4\delta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x.

$$x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}$$

$$\Rightarrow x = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

Bandwidth ωb in the frequency response is inversely proportional to the rise time tr in the time domain transient response.

Bode diagrams

The Bode plot or the Bode diagram consists of two plots −

- Magnitude plot
- Phase plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, y axis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB is -

$$M=20\log|G(j\omega)H(j\omega)|$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi=\angle G(j\omega)$$

Type of term	$G(j\omega)H(j\omega)$	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20 \log K$	0
Zero at origin	$j\omega$	20	$20 \log \omega$	90
'n' zeros at origin	$(j\omega)^n$	$20\,n$	$20\,n \log \omega$	$90\,n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20 \log \omega$	$-90 \ or \ 270$
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20\,n$	$-20\,n \log \omega$	$-90\,n \ or$ $270\,n$
Simple zero	$1 + j\omega r$	20	$0 \ for \ \omega < \frac{1}{r}$ $20 \ \log \omega r \ for \ \omega > \frac{1}{r}$	$0 \ for \ \omega < \frac{1}{r}$ $90 \ for \ \omega$ $> \frac{1}{r}$
Simple pole	$\frac{1}{1+j\omega r}$	-20	$0 \ for \ \omega < \frac{1}{r}$ $-20 \ \log \omega r \ for \ \omega > \frac{1}{r}$	$0 \ for \ \omega < \frac{1}{r}$ $-90 \ or \ 270$ $for \ \omega > \frac{1}{r}$
Second order derivative term	$\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n} \right)$	40	$40 \ \log \omega_n \ for \ \omega < \omega_n$ $20 \ \log \left(2\delta\omega_n^2 \right) \ for \ \omega$ $= \omega_n$ $40 \ \log \omega \ for \ \omega > \omega_n$	$0 \ for \ \omega$ $< \omega_n$ $90 \ for \ \omega$ $= \omega_n$ $180 \ for \ \omega$ $> \omega_n$

| Second order integral term | $\dfrac{1}{\omega_n^2\left(1-\dfrac{\omega^2}{\omega_n^2}+\dfrac{2j\delta\omega}{\omega_n}\right)}$ | -40 | $-40\ \log\ \omega_n\ for\ \omega<\omega_n$

 $-20\ \log\ (2\delta\omega_n^2)\ for\ \omega$ $=\omega_n$

 $-40\ \log\ \omega\ for\ \omega>\omega_n$ | $-0\ for\ \omega$ $<\omega_n$

 $-90\ for\ \omega$ $=\omega_n$

 $-180\ for\ \omega$ $>\omega_n$ |

Consider the open loop transfer function G(s)H(s)=KG(s)H(s).

Magnitude M=20logK dB

Phase angle ϕ=0 degrees

If K=1, then magnitude is 0 dB.

If K>1, then magnitude will be positive.

If K<1, then magnitude will be negative.

The following figure shows the corresponding Bode plot.
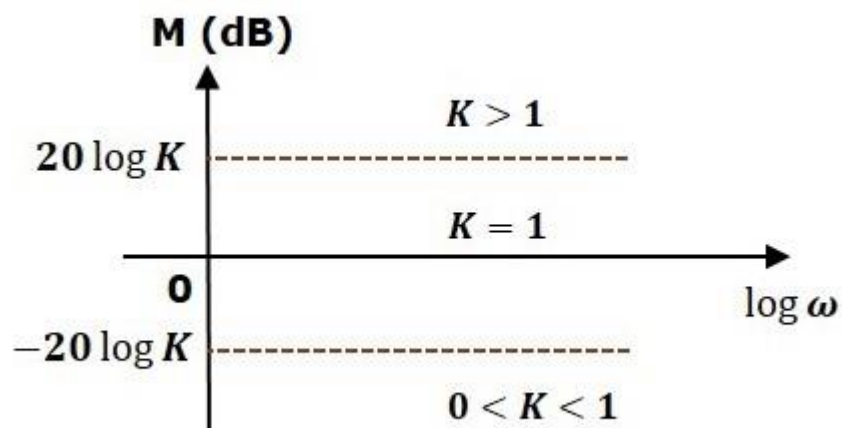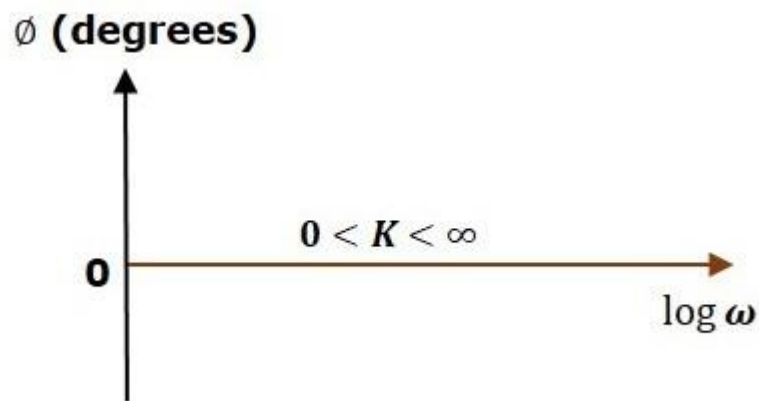
The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift 20logK dB above the 0 dB line. For the negative values of K, the horizontal line will shift 20logK dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.

Consider the open loop transfer function G(s)H(s)=s.

Magnitude M=20logωdB

Phase angle ϕ=90

At ω=0.1 rad/sec, the magnitude is -20 dB.

At ω=1 rad/sec, the magnitude is 0 dB.

At ω=10 rad/sec, the magnitude is 20 dB.

The following figure shows the corresponding Bode plot.
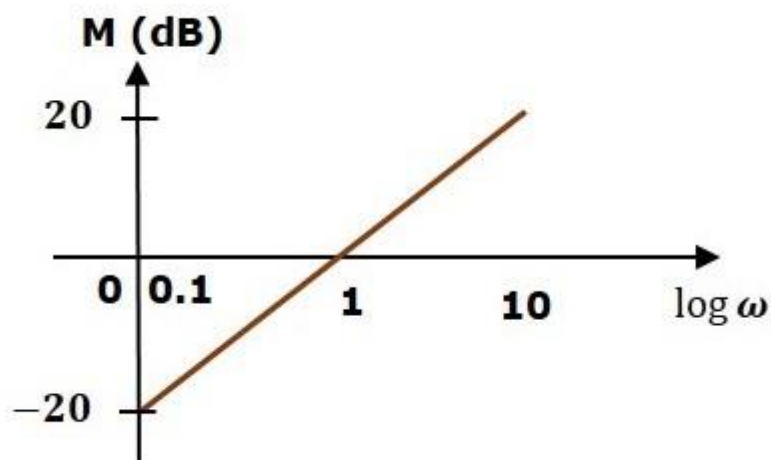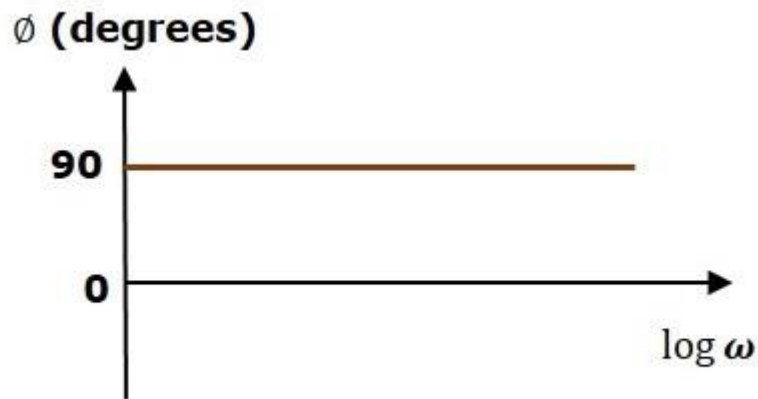
The magnitude plot is a line, which is having a slope of 20 dB/dec. This line started at $\omega=0.1$ rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at $\omega=1$ rad/sec. In this case, the phase plot is 90^0 line.

Consider the open loop transfer function G(s)H(s)=1+sτ.

Magnitude $M=20\log\sqrt{1+\omega^2\tau^2}$ dB

Phase angle $\phi=\tan^{-1}\omega\tau$ degrees

For $\omega<\frac{1}{\tau}$, the magnitude is 0 dB and phase angle is 0 degrees.

For $\omega>\frac{1}{\tau}$, the magnitude is $20\log\omega\tau$ dB and phase angle is 90^0.

The following figure shows the corresponding Bode plot.
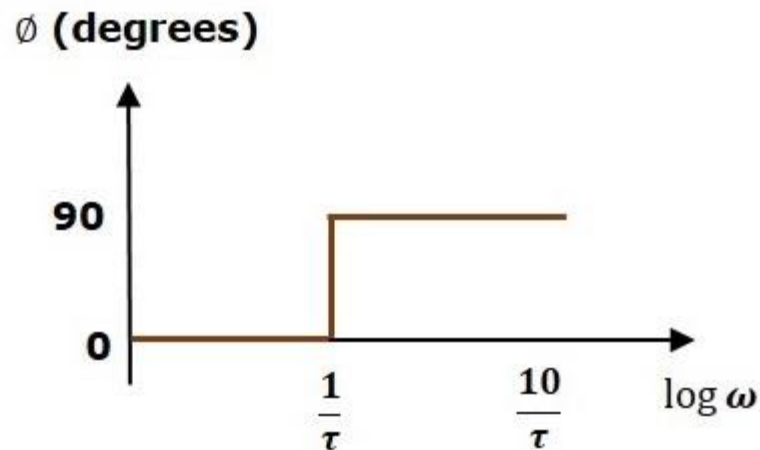
The magnitude plot is having magnitude of 0 dB up to ω=1τ rad/sec. From ω=1τ rad/sec, it is having a slope of 20 dB/dec. In this case, the phase plot is having phase angle of 0 degrees up to ω=1 rad/sec and from here, it is having phase angle of 90^0. This Bode plot is called the **asymptotic Bode plot**.

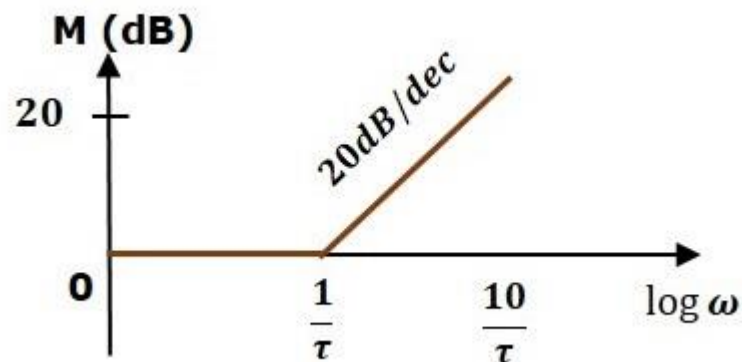As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Rules for Construction of Bode Plots

Follow these rules while constructing a Bode plot.

- Represent the open loop transfer function in the standard time constant form.

- Substitute, s=jω in the above equation.

- Find the corner frequencies and arrange them in ascending order.

- Consider the starting frequency of the Bode plot as $1/10^{th}$ of the minimum corner frequency or 0.1 rad/sec whichever is smaller value and draw the Bode plot upto 10 times maximum corner frequency.

- Draw the magnitude plots for each term and combine these plots properly.

- Draw the phase plots for each term and combine these plots properly.

Determination of frequency domain specifications and Phase margin and Gain margin - Stability analysis from Bode Plots

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

The frequency at which the phase plot is having the phase of -180^0 is known as **phase cross over frequency**. It is denoted by ωpc. The unit of phase cross over frequency is **rad/sec**.

Gain Cross over Frequency

The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by ωgc. The unit of gain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

- If the phase cross over frequency ωpc is greater than the gain cross over frequency ωgc,then the control system is **stable**.
- If the phase cross over frequency ωpcωpc is equal to the gain cross over frequency ωgc, then the control system is **marginally stable**.
- If the phase cross over frequency ωpc is less than the gain cross over frequency ωgc, then the control system is **unstable**.

Gain Margin

Gain margin GM is equal to negative of the magnitude in dB at phase cross over frequency.

$$GM = 20\log\left(\frac{1}{M_{pc}}\right) = 20logM_{pc}$$

Where, Mpc is the magnitude at phase cross over frequency. The unit of gain margin (GM) is **dB**.

Phase Margin

The formula for phase margin PM is

$$PM = 180^0 + \phi_{gc}$$

Where, ϕgc is the phase angle at gain cross over frequency. The unit of phase margin is **degrees**.

The stability of the control system based on the relation between gain margin and phase margin is listed below.

- If both the gain margin GM and the phase margin PM are positive, then the control system is **stable**.
- If both the gain margin GM and the phase margin PM are equal to zero, then the control system is **marginally stable**.
- If the gain margin GM and / or the phase margin PM are/is negative, then the control system is **unstable**.

Polar Plots

Polar plot is a plot which can be drawn between magnitude and phase. Here, the magnitudes are represented by normal values only.

The polar form of G(jω)H(jω) is

G(jω)H(jω)=|G(jω)H(jω)|∠G(jω)H(jω)

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of G(jω)H(jω)by varying ωω from zero to ∞. The polar graph sheet is shown in the following figure.
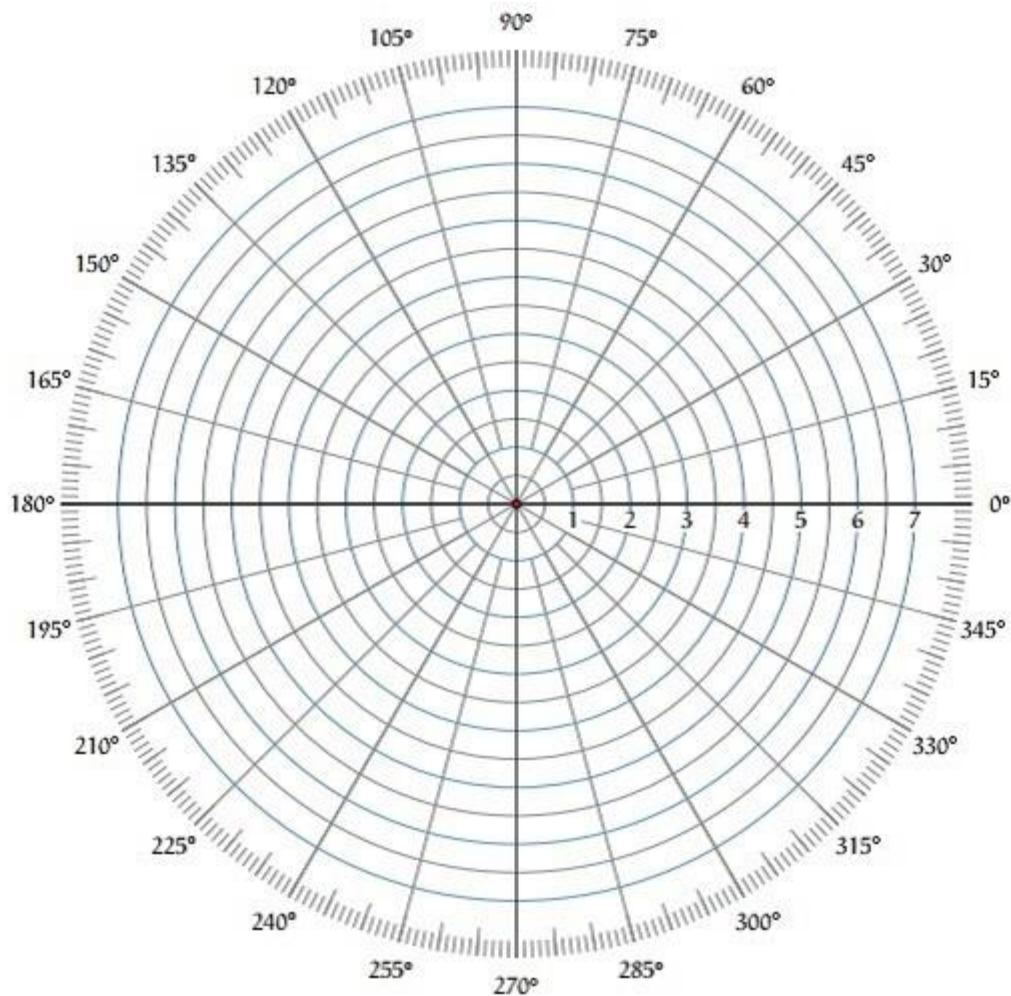
This graph sheet consists of concentric circles and radial lines. The **concentric circles** and the **radial lines** represent the magnitudes and phase angles respectively. These angles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0 in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

- Substitute, s=jω in the open loop transfer function.
- Write the expressions for magnitude and the phase of G(jω)H(jω).
- Find the starting magnitude and the phase of G(jω)H(jω) by substituting ω=0. So, the polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of G(jω)H(jω) by substituting ω=∞. So, the polar plot ends with this magnitude and the phase angle.

- Check whether the polar plot intersects the real axis, by making the imaginary term of G(jω)H(jω) equal to zero and find the value(s) of ω.
- Check whether the polar plot intersects the imaginary axis, by making real term of G(jω)H(jω) equal to zero and find the value(s) of ω.
- For drawing polar plot more clearly, find the magnitude and phase of G(jω)H(jω) by considering the other value(s) of ω.

Nyquist Plots

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from −∞ to ∞. That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

The Nyquist stability criterion works on the **principle of argument**. It states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding G(s)H(s) plane must encircle the origin P−Z times. So, we can write the number of encirclements N as,

$$N=P-Z$$

- If the enclosed 's' plane closed path contains only poles, then the direction of the encirclement in the G(s)H(s) plane will be opposite to the direction of the enclosed closed path in the 's' plane.
- If the enclosed 's' plane closed path contains only zeros, then the direction of the encirclement in the G(s)H(s) plane will be in the same direction as that of the enclosed closed path in the 's' plane.

Let us now apply the principle of argument to the entire right half of the 's' plane by selecting it as a closed path. This selected path is called the **Nyquist** contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half of the 's' plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

- The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.

- The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

We know that the open loop control system is stable if there is no open loop pole in the the right half of the 's' plane.

i.e.,$P=0 \Rightarrow N=-Z$

We know that the closed loop control system is stable if there is no closed loop pole in the right half of the 's' plane.

i.e.,$Z=0 \Rightarrow N=P$

Nyquist stability criterion states the number of encirclements about the critical point (1+j0) must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane. The shift in origin to (1+j0) gives the characteristic equation plane.

Rules for Drawing Nyquist Plots

Follow these rules for plotting the Nyquist plots.

- Locate the poles and zeros of open loop transfer function G(s)H(s) in 's' plane.
- Draw the polar plot by varying ωω from zero to infinity. If pole or zero present at s = 0, then varying ωω from 0+ to infinity for drawing polar plot.
- Draw the mirror image of above polar plot for values of ωω ranging from −∞ to zero (0⁻ if any pole or zero present at s=0).

- The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point (-1+j0) lies outside the encirclement, then the closed loop control system is absolutely stable.

Stability Analysis using Nyquist Plots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180^0) is known as the **phase cross over frequency**. It is denoted by ωpc.

Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the **gain cross over frequency**. It is denoted by ωgc

The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

- If the phase cross over frequency ωpc is greater than the gain cross over frequency ωgc, then the control system is **stable**.
- If the phase cross over frequency ωpc is equal to the gain cross over frequency ωgc, then the control system is **marginally stable**.
- If phase cross over frequency ωpc is less than gain cross over frequency ωgc, then the control system is **unstable**.

Gain Margin

The gain margin GM is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

Where, Mpc is the magnitude in normal scale at the phase cross over frequency.

Phase Margin

The phase margin PM is equal to the sum of 180^0 and the phase angle at the gain cross over frequency.

$$PM = 180^0 + \phi_{gc}$$

Where, ϕgc is the phase angle at the gain cross over frequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

- If the gain margin GM is greater than one and the phase margin PM is positive, then the control system is stable.
- If the gain margin GM is equal to one and the phase margin PM is zero degrees, then the control system is marginally stable.
- If the gain margin GM is less than one and / or the phase margin PM is negative, then the control system is unstable.

Compensation Techniques

Lag Controller design in frequency Domain

The transfer function of a phase-lag controller is given by

$$G_{lag}(j\omega) = \left(\frac{p_1}{z_1}\right)\frac{z_1 + j\omega}{p_1 + j\omega} = \frac{1 + j\frac{\omega}{z_1}}{1 + j\frac{\omega}{p_1}}, \quad z_1 > p_1$$
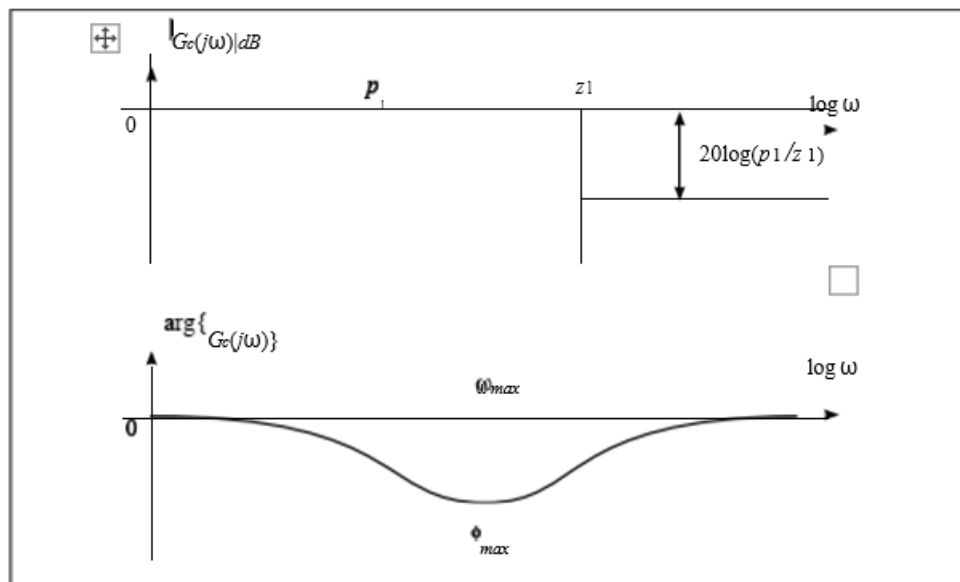
Due to attenuation of the phase-lag controller at high frequencies, the frequency bandwidth of the compensated system (controller and system in series) is reduced. Thus, the phase-lag controllers are used in order to decrease the system bandwidth (to slow down the system response). In addition, they can be used to improve the stability margins (phase and gain) while keeping the steady state errors constant.

Expressions for ω^{max} and ϕ^{max} of a phase-lag controller will be derived in the next subsection in the context of the study of a phase-lead controller.

Lead Controller design in frequency Domain

The transfer function of a phase-lead controller is

$$G_{lead}(j\omega) = \left(\frac{p_2}{z_2}\right)\frac{z_2 + j\omega}{p_2 + j\omega} = \frac{1 + j\frac{\omega}{z_2}}{1 + j\frac{\omega}{p_2}}, \quad p_2 > z_2$$
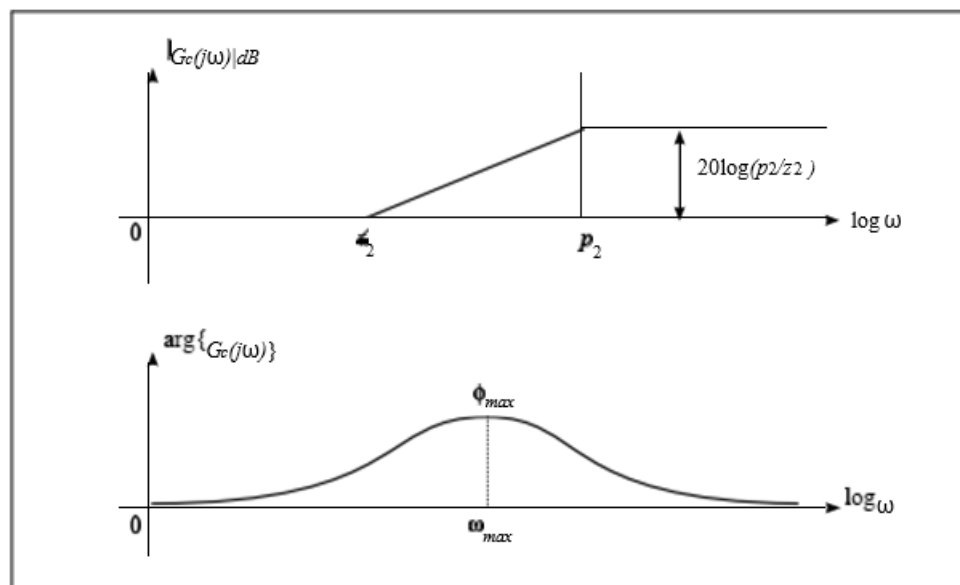
Due to phase-lead controller (compensator) amplification at higher frequencies, it increases the bandwidth of the compensated system. The phase-lead controllers are used to improve the gain and phase stability margins and to increase the system bandwidth (decrease the system response rise time).

Lead-Lag Controller design in frequency Domain

The phase-lag-lead controller has the features of both phase-lag and phase-lead controllers and can be used to improve both the transient response and steady state errors. The frequency transfer function of the phase-lag-lead controller is given by

$$G^{cd}(j\omega) = \frac{(j\omega + z_1)(j\omega + z_2)}{(j\omega + p_1)(j\omega + p_2)} = \frac{z_1 z_2}{p_1 p_2} \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)}$$

$$= \frac{\left(1 + j\frac{\omega}{z_1}\right)\left(1 + j\frac{\omega}{z_2}\right)}{\left(1 + j\frac{\omega}{p_1}\right)\left(1 + j\frac{\omega}{p_2}\right)}, \qquad z_1 z_2 = p_1 p_2, \qquad p_2 > z_2 > z_1 > p_1$$
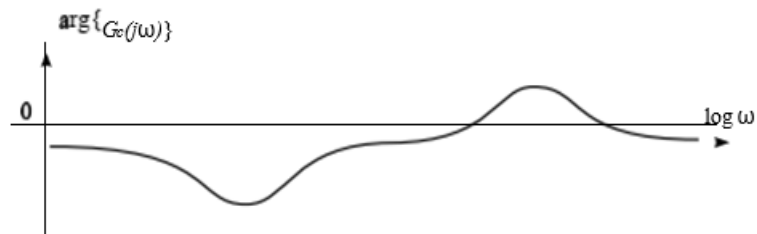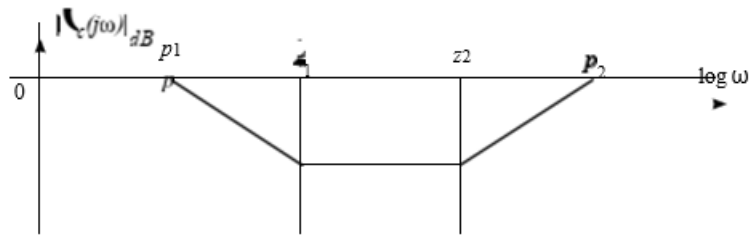
MODULE V

STATE SPACE ANALYSIS OF CONTINUOUS SYSTEMS

Concepts of state, state variables and state model

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,

$$X˙=AX+BU$$

$$Y=CX+DU$$

The first and the second equations are known as state equation and output equation respectively.

Where,

- X and X˙ are the state vector and the differential state vector respectively.

- U and Y are input vector and output vector respectively.

- A is the system matrix.

- B and C are the input and the output matrices.

- D is the feed-forward matrix.

Basic Concepts of State Space Model

The following basic terminology involved in this chapter.

State

It is a group of variables, which summarizes the history of the system in order to predict the future values (outputs).

State Variable

The number of the state variables required is equal to the number of the storage elements present in the system.

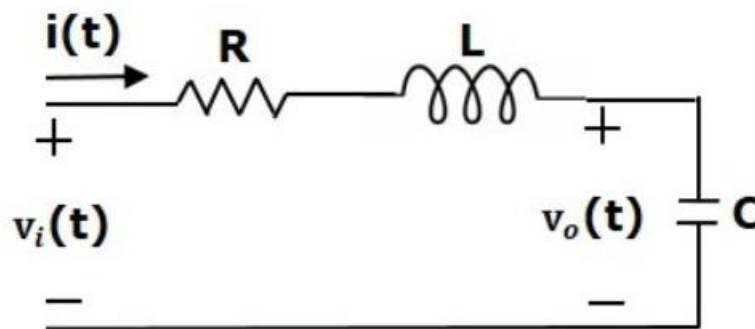**Examples** − current flowing through inductor, voltage across capacitor

State Vector

It is a vector, which contains the state variables as elements.

In the earlier chapters, we have discussed two mathematical models of the control systems. Those are the differential equation model and the transfer function model. The state space model can be obtained from any one of these two mathematical models. Let us now discuss these two methods one by one.

Derivation of state models from block diagrams

Consider the following series of the RLC circuit. It is having an input voltage, vi(t) and the current flowing through the circuit is i(t).

There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, i(t) and the voltage across capacitor, vc(t).

From the circuit, the output voltage, v0(t) is equal to the voltage across capacitor, vc(t),

$$v_0(t) = v_c(t)$$

Apply KVL around the loop

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$$

$$\Rightarrow \frac{di(t)}{dt} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

The voltage across the capacitor is –

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{dv_c(t)}{dt} = \frac{i(t)}{C}$$

State vector,
$$X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Differential state vector,
$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix}$$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,
$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, \ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \ and \ D = \begin{bmatrix} 0 \end{bmatrix}$$

Diagonalization - Solving the Time invariant state equations

Consider the two types of transfer functions based on the type of terms present in the numerator.

- Transfer function having constant term in Numerator.
- Transfer function having polynomial function of 's' in Numerator.

Transfer function having constant term in Numerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0 U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

Let,

$$y(t) = x_1$$

$$\frac{dy(t)}{dt} = x_2 = \dot{x}_1$$

$$\frac{d^2 y(t)}{dt^2} = x_3 = \dot{x}_2$$

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$$\frac{d^{n-1} y(t)}{dt^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{d^n y(t)}{dt^n} = \dot{x}_n$$

and u(t)=u

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1 x_2 + a_0 x_1 = b_0 u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1}x_n + b_0 u$$

The output equation is –

$$y(t) = y = x_1$$

The state space model is –

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here, $D = [0]$.

We know the state space model of a Linear Time-Invariant (LTI) system is –

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Apply Laplace Transform on both sides of the state equation.

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Apply Laplace Transform on both sides of the output equation.

$$Y(s) = CX(s) + DU(s)$$

Substitute, X(s) value in the above equation.

$$\Rightarrow Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$\Rightarrow Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

The above equation represents the transfer function of the system. So, we can calculate the transfer function of the system by using this formula for the system represented in the state space model.

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Example

Let us calculate the transfer function of the system represented in the state space model as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when D=[0] is –

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore, the transfer function of the system for the given state space model is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

State Transition Matrix and it's Properties

If the system is having initial conditions, then it will produce an output. Since, this output is present even in the absence of input, it is called **zero input response** $x_{ZIR}(t)$. Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At} X(0) = L^{-1}\left\{[sI - A]^{-1}X(0)\right\}$$

From the above relation, we can write the state transition matrix $\phi(t)$ as

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

So, the zero input response can be obtained by multiplying the state transition matrix $\phi(t)$ with the initial conditions matrix.

Following are the properties of the state transition matrix.

- If t=0, then state transition matrix will be equal to an Identity matrix.

$$\phi(0) = I$$

- Inverse of state transition matrix will be same as that of state transition matrix just by replcing 't' by '-t'.

$$\phi^{-1}(t) = \phi(-t)$$

- If t=t1+t2, then the corresponding state transition matrix is equal to the multiplication of the two state transition matrices at t=t1 and t=t2.

$$\phi(t_1 + t_2) = \phi(t_1)\phi(t_2)$$

Concepts of Controllability and observability

Controllability

A control system is said to be **controllable** if the initial states of the control system are transferred (changed) to some other desired states by a controlled input in finite duration of time.

We can check the controllability of a control system by using **Kalman's test**.

- Write the matrix Qc in the following form.

$$Q_c = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

- Find the determinant of matrix Qc and if it is not equal to zero, then the control system is controllable.

Observability

A control system is said to be **observable** if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

We can check the observability of a control system by using **Kalman's test**.

- Write the matrix Qo in following form.

$$Q_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \ldots & (A^T)^{n-1} C^T \end{bmatrix}$$

- Find the determinant of matrix Qo and if it is not equal to zero, then the control system is observable.

Example

Let us verify the controllability and observability of a control system which is represented in the state space model as,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \end{bmatrix}, D = [0] \quad and \quad n = 2$$

For n=2n=2, the matrix Qc will be

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix}$$

We will get the product of matrices A and B as,

$$AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_c| = 1 \neq 0$$

Since the determinant of matrix Qc is not equal to zero, the given control system is controllable.

For n=2, the matrix Qo will be –

$$Q_o = \begin{bmatrix} C^T & A^T C^T \end{bmatrix}$$

Here,

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We will get the product of matrices A^T and C^T as

$$A^T C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow |Q_o| = -1 \quad \neq 0$$

Since, the determinant of matrix Qo is not equal to zero, the given control system is observable.

Therefore, the given control system is both controllable and observable.